Topology and its Applications 41 (1991) 73-112
North-Holland

The topology of certain spaces of measures*

Charles Stegall

Institut für Mathematik, Johannes Kepler Universität, A-4040 Linz, Austria

Received 4 June 1990
Revised 24 August 1990

Abstract

Stegall, C., The topology of certain spaces of measures, Topology and its Applications 41 (1991) 73-112.

We continue the investigation of a class of topological spaces we introduced earlier. The main utility of this class is the study of Gâteaux differentiation in Banach spaces and, hence, the classification of Banach spaces. In particular, we consider noncompact spaces in this class, as well as spaces of measures on these spaces. We show how several other classes of spaces introduced by various authors are related to our class.

Keywords: Topology, measure theory, Banach spaces, Gâteaux differentiation.

AMS(MOS) Subj. Class.: Primary 46B99; secondary 54C60, 26B05.

1. Introduction

We continue the investigation we began in [32, 33] of a certain class C of topological spaces. We introduced this class to facilitate the study of Gâteaux differentiation of convex functions defined in Banach spaces (for much more about this, see [37, 35]) but there have also been applications in the isomorphic theory of Banach spaces, particularly the classification of $C(K)$ spaces (the Banach space of continuous functions on the compact space K). There have also been a number of recent papers in general topology studying related classes of spaces. Of course, the problems concerning Gâteaux differentiation are not purely topological in nature but what does help is information about the topological structure of the space of nonnegative measures on compact spaces. One of the techniques, that of minimal mappings, is very old [2]; we have even used this technique in Banach spaces as long ago as [14]. The class C may be briefly defined in the following way: a Hausdorff space T is in C if given any upper semicontinuous compact valued map Φ that is defined on a Baire space B with values in $\not{k}(T)$ then there exists a (single valued) function $f: B \to T$ such that $f(b) \in \Phi(b)$ for all $b \in B$ and f is continuous at each

* This material is based on lectures in Linz in the winter of 1987.

point of a dense G_δ subset of B. This is one of many equivalent definitions; see Section 4 and [33] for a discussion of the various definitions. In [35] we studied the class $\boldsymbol{K}$ of k-analytic spaces in $\boldsymbol{C}$. It is easy to see that $\boldsymbol{C}$ contains all metric spaces and it is proved in [35] that analytic spaces are in $\boldsymbol{K}$. Suppose that X is a Banach space such that X^* in the weak* topology is in $\boldsymbol{C}$, V is an open subset of some other Banach space Y, $g: V \to X$ is continuous, g is Gâteaux differentiable at each point of a dense G_δ of V and $f: X \to \mathbb{R}$ is continuous and convex; then the composition $f \circ g$ is Gâteaux differentiable at each point of a dense G_δ of V. The proof of this result is found in [35] where it is also shown that the derivative of $f \circ g$ cannot be obtained by a chain rule in the usual way. We obtain a number of topological results that may be of independent interest. Some of our results are repetitious but no proofs are repeated. These different proofs could be used to make finer determination of the various classes of topological spaces involved. For example, some of the results in Section 7 can be deduced from the formally more general setting of Section 8. We close with a discussion of various examples that show the interaction of these ideas between functional analysis and topology.

2. Definitions and notation

For any undefined notions see [7, 17, 35]. All spaces considered are Hausdorff. Many results do not require any regularity assumptions and in some cases we specifically mention that we do not assume complete regularity. When dealing with measures we assume that the spaces involved are completely regular. A Baire space is a space in which no nonempty open set is first category. A function $\Phi: B \to \wp(T)$, $\wp(T)$ is the power set of T, is upper semicontinuous if for any closed subset C of T,

$$\{b \in B: \Phi(b) \cap C \neq \emptyset\}$$

is closed; an usc compact valued map is minimal if it does not properly contain (as a relation in $B \times T$) any other usc compact valued map. We always assume that each $\Phi(b)$ is not empty. We use without stating the well-known and trivial duality between perfect maps and usc compact valued maps (if more information is really required see [2] or [33]). We shall constantly use the following equivalences for an usc compact valued map $\Phi: B \to \wp(T)$:

(i) Φ is minimal;

(ii) for each closed subset F of T and each open subset U of B so that $\Phi(b) \cap F \neq \emptyset$ for each $b \in U$ we have that $\Phi(U) = \bigcup \{\Phi(b): b \in U\} \subseteq F$;

(iii) for any $b \in B$, any $t \in \Phi(b)$, any open sets U and V with $b \in U$ and $t \in V$ there exists a nonempty open subset W of U so that $\Phi(W) \subseteq V$.

The above equivalences are contained in the following proposition.

Proposition 2.1. *Suppose that $p: T \to S$ is perfect onto. Then p is minimal if and only if for any open sets $U \subseteq T$ and $W \subseteq S$ such that $p(U) \cap W \neq \emptyset$ there exists an open set V such that $\emptyset \neq V \subseteq p(U) \cap W$ and $p^{-1}(V) \subseteq U$.*

Proof. If p has this property and $\emptyset \neq U \subseteq T$ is open then $p^{-1}(\operatorname{int} p(U)) \neq \emptyset$. This proves that $p(T \setminus U) \neq S$ and that p is minimal. Suppose that $p(U) \cap W \neq \emptyset$. Observe that $U \setminus p^{-1}(S \setminus W) \neq \emptyset$. Thus

$$(T \setminus U) \cup p^{-1}(S \setminus W) \neq T$$

and

$$V = S \setminus ((p(T \setminus U) \cup (S \setminus W)) = W \cap (S \setminus p(T \setminus U))$$

is the desired set. □

An immediate consequence is the following:

Lemma 2.2. *Suppose that p is a minimal perfect map from T to S, $F \subseteq T$ closed and U is an open subset of S such that $U \subseteq p(F)$. Then*

$$p^{-1}(\operatorname{int} \bar{U}) \subseteq F.$$

Let T be a topological space and $\mathcal{U}$ the nonempty open subsets of T. We shall say that T admits a tactic (precisely, an α-favorable tactic, see [3]) if there exists a function $s : \mathcal{U} \to \mathcal{U}$ so that $s(U) \subseteq U$ and if $\{U_n\}$ is any sequence in $\mathcal{U}$ so that $s(U_n) \supseteq U_{n+1}$ then $\emptyset \neq \bigcap_n U_n$.

Proposition 2.3. *Suppose that p is a minimal perfect map from T to S and that S admits a tactic s. Then T admits a tactic s'.*

Proof. If $\emptyset \neq U$ is an open subset of T define

$$s'(U) = p^{-1}(s(S \setminus p(T \setminus U))).$$

It is completely routine to check that this works. Suppose that

$$U_{n+1} \subseteq s'(U_n).$$

If we define

$$V_n = S \setminus p(T \setminus U_n)$$

then

$$V_n = S \setminus p(T \setminus U_n) \subseteq p(U_n) \subseteq s(V_{n-1}).$$

Since $\bigcap_n V_n \neq \emptyset$ we have that $\bigcap_n p^{-1}(V_n) \neq \emptyset$. Clearly, $p^{-1}(V_n) \subseteq s'(U_{n-1})$ and this completes the proof. □

Definition. If $S \subseteq T$ then S is an F-Souslin subspace of T if S can be obtained by applying the Souslin operation to closed subsets of T.

Definition. A completely regular space A is Čech complete if it satisfies any one of the following equivalent conditions:

(i) A is homeomorphic to a G_δ subset of a compact space;

(ii) A is G_δ in βA, the Stone-Čech compactification of A;

(iii) A is G_δ in any space (completely regular or not) in which it is densely embedded.

We require the well-known result that the following are equivalent for a space T:

(i) T is homeomorphic to a complete metric space;

(ii) T is homeomorphic to a G_δ subset of a complete metric space;

(iii) T is metric and Čech complete;

(iv) T is metric and if $i: T \to M$ is any embedding into a metric space M then $i(T)$ is G_δ in M.

Definition. If T is a topological space define

- $\boldsymbol{A}(T)$: the smallest algebra of subsets of T containing the closed sets;
- $\boldsymbol{B}(T)$: the Borel subsets of T;
- $\boldsymbol{C}(T)$: the Čech complete subsets of T;
- $\boldsymbol{D}(T)$: the smallest σ-algebra of subsets of T stable under the Souslin operation that contains $\boldsymbol{B}(T)$ and
- $\boldsymbol{BP}(T)$: the smallest σ-algebra of subsets of T that contains $\boldsymbol{B}(T)$ and the first category sets.

The theorem of Szpilrajn-Marczewski says that $\boldsymbol{BP}(T)$, the Baire property sets, is stable under the Souslin operation (see [35] for a proof) and each $S \in \boldsymbol{BP}(T)$ has the representation $S = U \triangle F$ where U is open and F is first category.

We make the following observation concerning $\boldsymbol{A}(T)$: given any finite subset $\boldsymbol{F}$ of $\boldsymbol{A}(T)$ there exists a finite subalgebra $\boldsymbol{A}$ of $\boldsymbol{A}(T)$ such that each atom of $\boldsymbol{A}$ is the intersection of an open and a closed set and $\boldsymbol{F} \subseteq \boldsymbol{A}$. An easy induction argument on the cardinality of $\boldsymbol{F}$ verifies this.

Proofs of the following can be found in [34, 19].

Lemma 2.4. *Let B be a Baire space, $\{U_\alpha : \alpha \in \Gamma\}$ a collection of open sets whose union is dense in B and let $\{N_\alpha : \alpha \in \Gamma\}$ be a collection of first category sets. Then $\bigcup_\alpha U_\alpha \setminus N_\alpha$ contains a dense G_δ subset of B.*

If T is a topological space a Radon measure on T is a measure on $\boldsymbol{B}(T)$ that is inner regular with respect to compact sets.

3. Čech analytic spaces

The main definition and all of the results of this section are due to Fremlin [9]. The class of Čech analytic spaces provides a very good setting for our results.

Definition 3.1. A topological space T is Čech analytic if there exist a compact space K, a Polish space M and $S \subseteq K \times M$ such that S is Čech complete and T is homeomorphic to $\operatorname{proj}_K(S)$.

We shall not give proofs to the following results. Those readers not acquainted with [9] should take Theorem 3.2(iv) as the definition, which is usually our approach to Čech analytic spaces.

Theorem 3.2. *Let K be compact and $T \subseteq K$. The following are equivalent:*

(i) *T is Čech analytic;*

(ii) *there exists a Čech complete $S \subseteq K \times \mathbb{N}^{\mathbb{N}}$ such that* $\mathrm{proj}_K(S) = T$;

(iii) *there exists a Borel set $S \subseteq K \times \mathbb{N}^{\mathbb{N}}$ such that* $\mathrm{proj}_K(S) = T$;

(iv) *there exists a Souslin scheme*

$$\{S(\xi|k): k \in \mathbb{N}, \xi \in \mathbb{N}^{\mathbb{N}}\}$$

such that each $S(\xi|k)$ is an open or a closed subset of K and

$$T = \bigcup_{\xi} \bigcap_{k} S(\xi|k).$$

Theorem 3.3. *The following spaces are all Čech analytic:*

(i) *completely regular k-analytic spaces;*

(ii) *Čech complete spaces;*

(iii) *any Borel subspace of a Čech analytic space;*

(iv) *a subspace of a completely regular space obtained by applying the Souslin operation to a Souslin scheme of Čech analytic spaces.*

Theorem 3.4. *Čech analytic spaces are universally measurable.*

4. The class C

The following result is from [32, 33] where the utility of (iii) is shown in connection with differentiation problems.

Theorem 4.1. *Let B be a Baire space and T any space. The following are equivalent:*

(i) *any minimal usc compact valued map Φ defined on B with values in T is point valued at each point of a dense G_δ subset of B;*

(ii) *given any usc compact valued map Φ defined on B with values in T there exists a function $\lambda : B \to T$ such that λ is continuous at each point of a dense G_δ subset of B and $\lambda(b) \in \Phi(b)$ for all $b \in B$;*

(iii) *suppose that we are given topological spaces C, S and R, $C \subseteq T \times R$ and continuous functions $p : C \to S$ and $g : B \to S$ so that p is perfect onto; then there exists a function $\lambda : B \to T$ such that λ is continuous at each point of a dense G_δ subset of B and*

$$(\{\lambda(b)\} \times R) \cap C \neq \emptyset \quad \textit{and}$$

$$p((\{\lambda(b)\} \times R) \cap C) = \{g(b)\}.$$

For a sharper version of this see 4.4 of [35].

Theorem 4.2. *Let B be a Baire space and $\Phi: B \to \mathscr{k}(T)$ an usc compact valued map. Suppose that A is an F-Souslin subset of T. Then*

(i) *the set $\{b: \Phi(b) \cap A \neq \emptyset\}$ is a Baire property subset of B (it is of the form $U \triangle F$ where U is open and F is first category);*

(ii) *there exists an usc compact valued map $\psi: \bar{U} \to \mathscr{k}(T)$ such that for all b in a dense G_δ subset of U we have that $\psi(b) \subseteq \Phi(b) \cap A$;*

(iii) *if A is in C and Φ is minimal, then $\Phi(b)$ is a one point set for each b in a dense G_δ subset of U.*

Theorem 4.3 ("Charting", [34]). *Let X be a Banach space and Φ a minimal usc compact valued map (with respect to the weak* topology) from the Baire space B to X^*. Define for each $\delta > 0$,*

$$\begin{aligned} U_\delta = \{ W \subseteq B:\ & W \text{ is open, there exists } K_W \subseteq X^* \text{ so that} \\ & K_W \text{ is weak* } k\text{-analytic, } K_W \in C \text{ and} \\ & \{b \in W: \Phi(b) \cap (K_W + B(0, \delta)) = \emptyset\} \text{ is first category in } W\}. \end{aligned}$$

Suppose that for each $\delta > 0$, $U_\delta = \bigcup \{U \in U_\delta\}$ is dense in B. Then there exists a dense G_δ subset G of B such that for each b in G, $\Phi(b)$ is a one point set.

5. Residual Ramsey sets

Observe that Propositions 5.1 to 5.3 do not require complete regularity.

Proposition 5.1. *Let B be a Baire space and Φ a minimal upper semi-continuous compact valued map into the space T. Define*

$$\begin{aligned} \boldsymbol{B} = \{E \subseteq T:\ & \text{there exists a dense } G_\delta \text{ subset } G \subseteq B \text{ such that} \\ & \Phi(b) \subseteq E \text{ or } \Phi(b) \cap E = \emptyset \text{ for } b \in G\}. \end{aligned}$$

Then $\boldsymbol{B}$ is a σ-algebra stable under the Souslin operation that contains the Borel subsets of T.

Proof. Clearly, $\boldsymbol{B}$ is closed under complementation. Suppose that E and F are in $\boldsymbol{B}$ and let G and H be dense G_δ subsets of B corresponding to E and, respectively, F. If b is in $G \cap H$ and D is in the smallest algebra of sets determined by E and F then $\Phi(b) \subseteq D$ or $\Phi(b) \cap D = \emptyset$. This proves that $\boldsymbol{B}$ is an algebra. Suppose that $\{E(\xi|n)\}$ is a Souslin scheme in $\boldsymbol{B}$. Define $G(\xi|n)$ to be a dense G_δ subset of B associated with $E(\xi|n)$. Let $G = \bigcap \{G(\xi|n): \xi, n\}$. Then if b is in G and $\Phi(b) \cap E \neq \emptyset$ where

$$E = \bigcup_{\xi} \bigcap_{n} E(\xi|n)$$

is the result of the Souslin operation then for some particular ξ

$$\Phi(b)\cap\left(\bigcap_n E(\xi|n)\right)\neq\emptyset.$$

It follows that

$$\Phi(b)\subseteq\bigcap_n E(\xi|n)\subseteq E.$$

This proves that if b is in G then $\Phi(b)\subseteq E$ or $\Phi(b)\cap E=\emptyset$. We have that $\boldsymbol{B}$ is an algebra stable under the Souslin operation hence it must be a σ-algebra. If E is a closed subset of T let

$$G=\{b\colon \Phi(b)\cap E=\emptyset\}\cup\operatorname{int}\{b\colon \Phi(b)\cap E\neq\emptyset\}.$$

Since Φ is minimal G is an open dense subset of B and for $b\in G$, $\Phi(b)\subseteq E$ or $\Phi(b)\cap E=\emptyset$. □

A subset E of T that is in $\boldsymbol{B}$ will be said to be residual Ramsey with respect to Φ.

Proposition 5.2. *With the hypothesis as above (Proposition 5.1) suppose that $f\colon T\to\mathbb{R}$ is a Borel function. Then $\{b\in B\colon f$ is constant on $\Phi(b)\}$ contains a dense G_δ subset of B.*

Proof. If f is a simple function this follows immediately from Proposition 5.1. Suppose $\{f_n\}$ is a sequence of simple Borel functions that converges pointwise to f. Define

$$G_n=\{b\in B\colon f_n \text{ is constant on } \Phi(b)\}.$$

Clearly, if b is in $\bigcap_n G_n$ then f is constant on $\Phi(b)$. □

Proposition 5.3. *With the hypothesis as above (Proposition 5.1), define*

$$\boldsymbol{A}=\{E\in\boldsymbol{B}\colon \{b\in B\colon \Phi(b)\cap E\neq\emptyset\} \text{ is a Baire property subset of } B\}.$$

Then $\boldsymbol{A}$ is a σ-algebra that contains the Borel subsets of T and is stable under the Souslin operation and for each $E\in\boldsymbol{A}$, $\{b\in B\colon \Phi(b)\cap E\neq\emptyset\}$ differs from $\{b\in B\colon \Phi(b)\subseteq E\}$ by a set of the first category.

Proof. For each E in $\boldsymbol{A}$ define $B(E)=\{b\in B\colon \Phi(b)\cap E\neq\emptyset\}$ and choose $G(E)$ a dense G_δ subset of B that fulfills the residual Ramsey condition with respect to E. Clearly, $\boldsymbol{A}$ contains the closed sets and is closed under countable unions. Suppose E is in $\boldsymbol{A}$; then

$$B\setminus B(E)\subseteq B(T\setminus E)\subseteq(B\setminus B(E))\cup(B\setminus G(E)).$$

Since $B\backslash B(E)$ is a Baire property set and $B\backslash G(E)$ is first category this proves that $T\backslash E$ is in A. Similarly, if E and F are in A then $E\cap F$ is in A because

$$B(E)\cap B(F)\cap G(E)\cap G(F)\subseteq B(E\cap F)\subseteq B(E)\cap B(F).$$

This proves that A is an algebra. Suppose that $\{E(\xi|n)\}$ is a Souslin scheme in A. Let

$$E=\bigcup_{\xi}\bigcap_{n} E(\xi|n).$$

The theorem of Szpilrajn–Marczewski says that

$$H=\bigcup_{\xi}\bigcap_{n} B(E(\xi|n))$$

is a Baire property subset of B. Let $G=\bigcap\{G(E(\xi|n)):\ \xi, n\}$. We claim that $H\cap G=B(E)\cap G$:

$$\begin{aligned}
&b\in H\cap G\\
&\quad \text{iff } b\in G \text{ and for some } \xi,\ b\in \textstyle\bigcap_n B(E(\xi|n))\\
&\quad \text{iff } b\in G \text{ and for some } \xi \text{ and all } n,\ \Phi(b)\cap E(\xi|n)\neq\emptyset\\
&\quad \text{iff } b\in G \text{ and for some } \xi,\ \Phi(b)\subseteq \textstyle\bigcap_n E(\xi|n)\\
&\quad \text{iff } b\in G \text{ and } \Phi(b)\subseteq E\\
&\quad \text{iff } b\in G \text{ and } b\in B(E).
\end{aligned}$$

Thus $B(E)=(H\cap G)\cup(B(E)\backslash G)$ is a Baire property set. □

Proposition 5.4. *Let T be a compact space and define $M^+(T)$ to be the space of finite, nonnegative and inner regular measures on T in the weak* topology. Define*

$$D=\{E\subseteq T:\ \kappa_E:M^+(T)\to\mathbb{R} \text{ defined by } \kappa_E(\mu)=\mu(E) \text{ is a Borel function on } M^+(T)\}.$$

Then D is a σ-algebra containing the Borel subsets of T.

Proof. If E is a closed subset of T, then the function $f:M^+(T)\to[0,1]$ defined by $f(\mu)=\mu(E)$ is upper semicontinuous. The elementary formula $\mu(E\backslash F)=\mu(E)-\mu(E\cap F)$ shows that D contains the algebra generated by the closed sets. Also, D is a monotone class. Thus D contains the σ-algebra generated by the closed sets (see [35]). □

We require the following lemmata later.

Lemma 5.5. *Let B be a Borel subset of the compact space T. Then $M^+(B)$, the nonnegative measures on T that are supported on B, is a Borel subset of $C(T)^*$, where $C(T)^*$ has the weak* topology.*

Proof. Define $f: M^+(T) \to \mathbb{R}^3$ by $f(\mu) = (\mu(B), \mu(T \setminus B), \|\mu\|)$. The first two coordinates are Borel measurable and the third coordinate is lower semicontinuous. Thus f is Borel measurable and

$$f^{-1}\{(r, 0, r): r \in \mathbb{R}\}$$

is a Borel set. □

Lemma 5.6. *Let μ be a measure on the compact space T, let B be a measurable subset of T and let $\{U_\alpha\}$ be a collection of open subsets of T such that $\{U_\alpha \cap B\}$ is a pairwise disjoint collection. Then*

$$\mu\left(\bigcup_\alpha (U_\alpha \cap B)\right) = \sum_\alpha \mu(U_\alpha \cap B).$$

Proof. Choose $\varepsilon > 0$ and a compact set K such that

$$K \subseteq \bigcup_\alpha (U_\alpha \cap B)$$

and

$$\mu(K) + \varepsilon > \mu\left(\bigcup_\alpha (U_\alpha \cap B)\right).$$

Some finite subcollection $\{U_i: i \leq n\}$ of $\{U_\alpha\}$ covers K. Thus,

$$\mu\left(\bigcup_\alpha (U_\alpha \cap B)\right) - \varepsilon < \mu(K) \leq \sum_i \mu(U_i \cap B)$$

$$\leq \sum_\alpha \mu(U_\alpha \cap B) \leq \mu\left(\bigcup_\alpha (U_\alpha \cap B)\right). \qquad \square$$

We know of at least two places in the literature where the preposterous claim is made that counterexamples, even compact ones, exist to the following theorem. This seems to have come about because some attempts to generalize the results in [32, 33] were based on an incomplete understanding of the results there.

Theorem 5.7. *Suppose that K is compact, $T \in D(K)$ and T has the following property: if Φ is any minimal usc compact valued map with values in T defined on any Čech complete space B then Φ is point valued on a dense G_δ subset of B. It follows that if Φ is any minimal usc compact valued map with values in T defined on a completely regular Baire space B then Φ is point valued on a dense G_δ subset of B.*

Proof. Suppose that Φ is a minimal usc compact valued map defined on the Baire space B with values in T. We may extend Φ to a minimal usc compact valued map $\tilde{\Phi}$ defined on βB with values in K in a unique way. Define

$$G = \{(b, t): b \in B \text{ and } t \in \Phi(b)\}.$$

Define $\tilde{G}$ to be the closure of G in $\beta B \times K$. For $\tilde{b}$ in βB define

$$\tilde{\Phi}(\tilde{b}) = \mathrm{proj}_{\beta B}((\{\tilde{b}\} \times K) \cap \tilde{G}).$$

It is routine to check that $\tilde{\Phi}$ is a minimal usc compact valued map with the property that for $b \in B$, $\tilde{\Phi}(b) = \Phi(b)$. Since $T \in \boldsymbol{D}(K)$,

$$G = \{\tilde{b}\colon \tilde{\Phi}(\tilde{b}) \cap T \neq \emptyset\}$$

is a Baire property subset of βB and differs from

$$H = \{\tilde{b}\colon \tilde{\Phi}(\tilde{b}) \subseteq T\}$$

by a set of the first category. Suppose $G = U \triangle N_1$ and $H = U \triangle N_2$ where U is open and N_1 and N_2 are both of the first category in βB. Since B is a Baire space and $B \subseteq H$ we have that U is dense in βB. Let C be a dense G_δ subset of $U \backslash (N_1 \cup N_2)$. Since C is Čech complete we have that $\tilde{\Phi}|C$, which takes values only in T, takes single point values on a dense G_δ subset D of C. Thus Φ is single valued on $D \cap B$, which is a dense G_δ subset of B. □

An interesting consequence of this result and the example in [43] is that if S is an uncountable discrete set then $C(\beta S)$ in the simple topology is not in $\boldsymbol{D}(K)$ for any compact K.

In the following we may replace the weak topology by the weak topology generated by the extreme points of the unit ball of the dual X^* (see [35]).

Theorem 5.8. *Let X be a Banach space and suppose that X in its weak topology is homeomorphic to an element of $\boldsymbol{D}(K)$ for some compact space K. Let Φ be a minimal usc compact valued map, with respect to the weak topology, defined on the completely regular Baire space B with values in X. Then Φ is single valued at each point of a dense G_δ subset of B.*

Proof. A well-known theorem of Namioka [22] (see also [39, 41]) says that this result is true if B is a Čech complete space. Applying Theorem 5.7 we have that there exists a dense G_δ subset G of B such that Φ is norm continuous at each point of G. If U is an open subset of B and the diameter of $\Phi(U \cap G)$ is no more than $\varepsilon > 0$ then the diameter of $\overline{\Phi(U \cap G)} \supseteq \Phi(U)$ is also no more than ε.

6. Dense metrizable subspaces

In the following lemma it is not required that T and S be completely regular.

Lemma 6.1. *Suppose that C is Čech complete and $p\colon C \to T$ is perfect onto and T is a dense subspace of S. Then T is a G_δ subset of S. It follows that S is a Baire space.*

Proof. Let $\beta C \backslash C = \bigcup_n C_n$ where each C_n is compact and $C_n \subseteq C_{n+1}$ and define

$$W_n = \{W\colon W \text{ is an open subset of } S \text{ and } \overline{p^{-1}(W)}^{\beta C} \cap C_n = \emptyset\}.$$

Fix n and let $W \neq \emptyset$ be an open subset of S and choose $t \in W \cap T$. Since $p^{-1}(t)$ is compact there exists an open subset U of βC such that $p^{-1}(t) \subseteq U$ and $\bar{U}^{\beta C} \cap C_n = \emptyset$. Since $p(C \setminus U)$ is a closed subset of T we have that $t \in W \setminus \overline{p(C \setminus U)}^S$ and

$$W \setminus \overline{p(C \setminus U)}^S \in \mathbf{W}_n.$$

We have proved that $T \subseteq \bigcap_n \cup \{W \in \mathbf{W}_n\}$. Suppose that

$$s \in \left(\bigcap_n \left(\bigcup \{W \in \mathbf{W}_n\}\right)\right) \setminus T.$$

Choose $W_n \in \mathbf{W}_n$ so that $s \in W_n$. Observe that

$$K = \bigcap_n \overline{p^{-1}(W_n)}^{\beta C}$$

is a compact subset of C. Since S is Hausdorff there exist disjoint open subsets W' and W'' of S such that $s \in W'$ and $p(K) \subseteq W''$. Choose

$$c_n \in p^{-1}\left(W' \cap \left(\bigcap_{m \leq n} W_m\right)\right).$$

The sequence $\{c_n\}$ has a cluster point $c_0 \in K$ which implies that $p(c_0) \in \overline{W'}$ which is a contradiction. If we let D be a minimal closed subset of C such that $p(D) = p(C)$, then D is also Čech complete and a minimal perfect image of a Baire space is a Baire space (see [7]). It follows that S is a Baire space. $\square$

The following result is due to a number of authors before 1967 (see [7, p. 422] for references and a proof).

Theorem 6.2. *Suppose that $f: T \to S$ is continuous, open and onto where T is Čech complete and S is paracompact. Then there exists a closed and G_δ subset C of T such that $p|C$ is perfect onto S.*

We require the following related result, which probably is as old as the result above. It is only required that C be completely regular, which is part of the definition of Čech complete. Observe that the set $D \cap F$ defined below is also a Čech complete space.

Theorem 6.3. *Suppose that C is Čech complete and $p: C \to T$ is perfect onto and T is a dense subspace of S. Suppose $g: S \to R$ is continuous, open and onto. Then there exist a G_δ subset D of C, a closed subset F of C and a dense G_δ subset H of R Such that $f: D \cap F \to H$ is perfect onto where $f(c) = (g \circ p)(c)$.*

Proof. Firstly, assume that p is minimal. It follows easily that both S and R must be Baire spaces. Let $\beta C \setminus C = \bigcup_n C_n$ where each C_n is compact and

$$\emptyset = C_0 \subseteq C_n \subseteq C_{n+1}$$

and define

$$\boldsymbol{W}_n = \{W: W \text{ is an open subset of } S \text{ and } \overline{p^{-1}(W)}^{\beta C} \cap C_n = \emptyset\}.$$

Each $\boldsymbol{W}_n$ is not empty because p is minimal (use Proposition 2.1). Define $\boldsymbol{U}_0 = \{S\}$ and choose $\boldsymbol{U}_n$ maximal with respect to the following properties:

(i) $\boldsymbol{U}_n \subseteq \boldsymbol{W}_n$;

(ii) $\{g(U): U \in \boldsymbol{U}_n\}$ is a pairwise disjoint collection;

(iii) if $U \in \boldsymbol{U}_{n+1}$ then there exists a $U' \in \boldsymbol{U}_n$ so that $C \cap \overline{p^{-1}(U)}^{\beta C} \subseteq p^{-1}(U')$.

(iv) $\boldsymbol{U}_{n+1}$ is subordinate to $\boldsymbol{U}_n$.

Suppose that there exists $n > 0$ so that $\bigcup\{g(U): U \in \boldsymbol{U}_n\}$ is not dense in R. Let $\emptyset \neq O$ be an open subset of R such that

$$O \cap \left(\bigcup\{g(U): U \in \boldsymbol{U}_n\}\right) = \emptyset.$$

We may assume that we have the minimal such n. There exists $U_1 \in \boldsymbol{U}_{n-1}$ so that $O \cap g(U_1) \neq \emptyset$. Choose $\emptyset \neq U_2 \in \boldsymbol{W}_n$ so that

$$U_2 \subseteq g^{-1}(O) \cap U_1.$$

From the minimality of p there exist non empty open sets $Q_1 \subseteq C$ and $U_3 \subseteq U_2$ such that $p(Q_1) = U_3 \cap T$. Choose $\emptyset \neq Q_2$ and $\overline{Q_2}^{\beta C} \cap C \subseteq Q_1$. Again from minimality, there exist non empty open sets $Q_3 \subseteq Q_2$ and $U_4 \subseteq U_3$ such that $p(Q_3) = U_4 \cap T$. It is easy to see that $\boldsymbol{U}_n \cup \{U_4\}$ satisfies (i) to (iv) which contradicts the maximality requirement. Let $D' = \bigcap_n \cup \boldsymbol{U}_n$ and let $H = g(D')$. We claim that

$$H = \bigcap_n \bigcup\{g(U): U \in \boldsymbol{U}_n\}.$$

Choose $U_n \in \boldsymbol{U}_n$ so that $U_{n+1} \subseteq U_n$. Remember that

$$\overline{p^{-1}(U_{n+1})}^{\beta C} \cap C \subseteq p^{-1}(U_n).$$

It follows that

$$\bigcap_n \overline{p^{-1}(U_n)}^{\beta C} = C \cap \left(\bigcap_n \overline{p^{-1}(U_n)}^{\beta C}\right)$$
$$\subseteq \bigcap_n p^{-1}(U_n) \subseteq \bigcap_n \overline{p^{-1}(U_n)}^{\beta C} = K$$

which is a compact subset of C and $p(K) \subseteq D'$. Suppose that

$$r \in \bigcap_n g(U_n) \backslash g(p(K)).$$

Choose O open so that $g(p(K)) \subseteq O$ and $r \notin \bar{O}$ and define $U = g^{-1}(O)$. It follows that $g(\bar{U}) \subseteq \bar{O}$. For all n we have that $U_n \backslash \bar{U} \neq \emptyset$. Choose $x_n \in p^{-1}(U_n \backslash \bar{U})$. The sequence $\{x_n\}$ has a cluster point x in K. Therefore, for large n, $p(x_n)$ is in U because $p(x)$ is in U. This is a contradiction. It remains only to show that $g|D'$ is a closed map. Suppose that F is a closed subset of S and $r \in H \backslash (g(F \cap D'))$ which means that

$$(F \cap D') \cap g^{-1}(r) = \emptyset.$$

Choose the unique $U_n \in \mathcal{U}_n$ so that $r \in \bigcap_n g(U_n)$. Then $E \subseteq D'$ is compact (the arguments of Lemma 6.1) where

$$E = \bigcap_n U_n \subseteq D'.$$

Then $E \cap F$ is compact and $r \notin g(E \cap F)$. Choose O open in R so that $g(E \cap F) \subseteq O$ and $r \notin \bar{O}$. As before define $U = g^{-1}(O)$. Then $E \cap F \subseteq U$ and arguments similar to those above show that $U_n \subseteq \bar{U}$ for some n. Since $g(\bar{U}) \subseteq \bar{O}$ and we have that

$$S = (S \setminus U_n) \cup g(\bar{U}),$$

$$F \cap D' \subseteq (D' \setminus U_n) \cup g(\bar{U}),$$

and

$$g(F \cap D') \cap g(U_n) \cap (R \setminus \bar{O}) = \emptyset.$$

The open set $g(U_n) \cap (R \setminus \bar{O})$ contains r and we have shown that $g|D'$ is a closed map. It remains only to define $D = p^{-1}(D')$ and $f = (g \circ p)|D$. The general case is obtained by restricting p to F where F is any minimal closed set such that $p(F) = T$. □

Corollary 6.4. *With everything the same as above except that we make the additional assumption that C is a complete metric space. Then we may find f, D and H as above with the additional property that f is a homeomorphism. It follows that R has a dense G_δ subset homeomorphic to a complete metric space.*

Proof. Repeat the construction above with the additional assumption that for each $U \in \mathcal{U}_n$ we have that the diameter of $p^{-1}(U)$ is less than 2^{-n} where the diameter is in a complete metric on C. This can also be deduced from the above by recalling that metric spaces are in $\mathcal{C}$. □

Theorem 6.5. *Suppose that C is Čech complete, T is a dense subspace of X, S is in $\mathcal{C}$ and we have the following continuous functions:*

$g: S \to R$ *open and onto,*

$q: S \to X$ *minimal perfect onto and*

$p: C \to T$ *perfect onto.*

Then there exists a subspace D of S such that $q|D$ is a homeomorphism onto a G_δ subspace of T and $g|D$ is a perfect map onto $g(D)$, which is a dense G_δ subspace of R.

Proof. Since S is in $\mathcal{C}$ there exists a subspace Y of S such that $q|Y$ is a homeomorphism onto a dense G_δ subset of T. It follows that Y is also the perfect image of a Čech complete space and since q is minimal we have that Y is dense in S. The result follows from the theorem above. □

The multivalued version is only diagram chasing.

Proposition 6.6. *Suppose S and T are Hausdorff spaces, $C \subseteq T$ is dense in T and C is Čech complete and Φ is a minimal usc compact valued map defined on T with values in S with the following properties:*

(i) *$\Phi(t)$ is a point for each $t \in G$ where G is a dense G_δ subset of T and*

(ii) *$\Phi(W) = \bigcup_{t \in W} \Phi(t)$ is open for each open subset W of T.*

Then there exist a G_δ subset D of C, a dense G_δ subset E of $\Phi(T)$ and $f: D \to E$ that is perfect and onto where $\{f(t)\} = \Phi(t)$.

One of the many very special cases of the above can be found in [4]. The following result is a folk theorem, and there exist a number of related results in the literature. It is, for example, an immediate consequence of the results above.

Proposition 6.7. *Let K be a compact Hausdorff space. The set*

$$G = \{f \in C(K): f \text{ attains its supremum at exactly one point of } K\}$$

contains a dense G_δ (in the norm topology) subset of $C(K)$ if and only if K contains a dense subspace that is homeomorphic to a complete metric space.

Proof. If we define $G = \{(f, k): f$ attains its supremum at $k\}$, the following elementary observations are classical and a complete proof is an easy consequence of them:

- $\mathrm{proj}_{C(K)}: G \to C(K)$ is minimal perfect onto,
- $\mathrm{proj}_K : G \to K$ is open onto and
- $\mathrm{proj}_K | \{(f, k): f$ attains its supremum only at $k\}$ is a homeomorphism. □

Corollary 6.8. *With the same hypothesis as in Proposition 6.6, if C is in $\mathbf{C}$ then E is in $\mathbf{C}$.*

We recall the following result from [38].

Theorem 6.9. *Suppose that K is a compact space and T is a dense subspace that admits a tactic s in the relative topology. Then the set*

$$\{f \in C(K): f \text{ attains its supremum on } T\}$$

contains a dense G_δ subset of $C(K)$.

We remark that a completely regular space in $\boldsymbol{K}$ satisfies the hypothesis of Definition 6.10. The following generalizes the results of [38].

Definition 6.10. We shall define $\mathbf{C}'$ to be the class of Hausdorff, but not necessarily completely regular, spaces such that A is in $\mathbf{C}'$ if there exist a compact space K and $S \in \mathbf{D}(K)$ so that S is in $\mathbf{C}$ and A is the perfect image of S.

Since C is stable under perfect images we have that C' is a subclass of C.

Theorem 6.11. *Let A be in C' and let T be any subspace of A. The following are equivalent:*

(i) *T admits a tactic;*
(ii) *T has a dense strongly countably complete subspace (see [10]);*
(iii) *T has a dense subspace that is Čech complete;*
(iv) *T has a dense subspace homeomorphic to a complete metric space.*

Proof. It is known that if T satisfies (ii) then T admits a tactic (see [35, 7.32]); (iv) implies (iii) and (iii) implies (ii) are elementary. Suppose that K is compact, $E \in \boldsymbol{D}(K)$ and $p: E \to A$ is perfect onto. If we replace A by $\bar{T}$ and E by a minimal closed subset F of $p^{-1}(\bar{T})$ such that $p(F) = \bar{T}$ then we may assume that T is dense in A and p is minimal. Under these assumptions, $p^{-1}(T)$ also has a tactic and we know that if $p^{-1}(T)$ has a dense subspace homeomorphic to a complete metric space so does T. To finish the proof we may assume that $T \subseteq A$ is dense and $A \subseteq K$ is dense. If we define $G = \{(f, k): f$ attains its supremum at $k\}$ and define $\Phi: C(K) \to \not{k}(K)$ by $\Phi(f) = \{K: f$ attains its supremum at $k\}$, then Φ is a minimal usc compact valued map. Combining previous results we have that

- $\{f \in C(K)$: $\Phi(f) \cap T \neq \emptyset\}$ contains a dense G_δ subset of $C(K)$ and
- $\{f \in C(K)$: $\Phi(f) \cap A \neq \emptyset$ and $\Phi(f) \cap (K \backslash A) \neq \emptyset\}$ is first category.

Since A is in C we have that

$$\{f \in C(K): f \text{ attains its supremum at exactly one point of } T\}$$

contains a dense G_δ subset H of $C(K)$. From Proposition 6.6 above there exists a G_δ subset M of H and a homeomorphism of M onto a dense subspace of T. □

The same arguments yield the following: if $A \in C$, if there exists a compact space K such that A is homeomorphic to an element of $\boldsymbol{D}(K)$, if T has a tactic and if $h: T \to A$ is one to one, then T has a dense set of G_δ points.

Theorem 6.12. *Let X be a locally convex topological vector space and K a compact and convex subset of X. Let $E(K)$ denote the extreme points of K. Suppose there exist a space A in C such that $A \in \boldsymbol{D}(S)$ for some compact S and an into homeomorphism $f: E(T) \to A$. Then $E(T)$ contains a dense G_δ subset that is homeomorphic to a complete metric space.*

Theorem 6.13. *Let K be compact and $B \subseteq T \subseteq K$. Suppose that B is Baire, T is in C and $T \in \boldsymbol{D}(K)$. Then B contains a dense G_δ subset that is metrizable. More generally, if B is Baire, B is dense in K, T is in C and T is a Baire property subset of K then B contains a dense G_δ subset that is metrizable.*

7. The spaces of Gruenhage

Among other things, it is a consequence of the theorem of Amir and Lindenstrauss [1] that weakly compact sets satisfy Lemma 7.1. A standard paracompactness argument shows that metric spaces also satisfy this lemma.

Lemma 7.1. *Let T be a Hausdorff (but not necessarily completely regular) space. The following conditions are equivalent:*

(i) *there exists a sequence $\{C_n\}$ of subsets of T and a sequence of collections U_n of open subsets of T such that for each n, each $t \in C_n$ belongs to exactly one element of U_n and $\{U \cap C_n : n \in \mathbb{N},\ U \in U_n\}$ separates the points of T;*

(ii) *there exists a sequence $\{C_n\}$ of subsets of T such that each C_n admits a partition of unity $\{f_{n,\alpha} : \alpha \in \Gamma_n\}$ and the collection of functions $\{\chi_{C_n} f_{n,\alpha} : n \in \mathbb{N}$ and $\alpha \in \Gamma_n\}$ separates the points of T;*

(iii) *there exists a sequence $\{C_n\}$ of subsets of T and a sequence of continuous functions $\phi_n : C_n \to c_0(\Gamma_n)$, where $c_0(\Gamma_n)$ has the weak topology, such that $\chi_{C_n}\phi_n$ separates the points of T;*

(iv) *there exists a sequence $\{C_n\}$ of subsets of T and a sequence of collections U_n of open subsets of T such that for each n, each $t \in C_n$ belongs to only finitely many elements of U_n, and*

$$\{V \cap C_n : n \in \mathbb{N},\ V \text{ is the intersection of finitely many elements of } U_n\}$$

separates the points of T.

Proof. Clearly, (ii), (iii) and (iv) are immediate consequences of (i). Suppose that we have (iii). Let $\{f_{n,\alpha}\}$ be the coordinate functions of ϕ_n. We may assume that each $f_{n,\alpha} \geq 0$. For each pair of rational numbers $0 < r < s$ choose open sets $U_{n,\alpha,r,s}$ in T such that

$$U_{n,\alpha,r,s} \cap C_n = \{r < f_{n,\alpha} < s\}.$$

Define $C_{n,r,s} = C_n \cap (\bigcup_\alpha U_{n,\alpha,r,s})$. Clearly, the collections $\{C_{n,r,s} : n \in \mathbb{N},\ r$ and s rational$\}$ and $\{U_{n,\alpha,r,s} : \alpha \in \Gamma_n\}$ satisfy the conditions of (iv). The remainder of the proof is contained in the decomposition method of [12]. Let U be a collection of open subsets of T. For each $p = 0, 1, \ldots$ define $J(U, p)$ to be those points of T that are in exactly p distinct elements of U. Define U^p to be that collection of open subsets of T consisting of intersections of p distinct elements of U. Observe that $\bigcup_{0 \leq p \leq q} J(U, p)$ is a closed subset of T for each q and it follows that each $J(U, p)$ is the intersection of a closed and an open set. Also, the collection of sets

$$\{J(U, p) \cap V : V \in U^p\}$$

is pairwise disjoint. The collections satisfying (i) are $\{J(U_m, p)\}$ and U_m^p.

Implicit in the hypothesis of Lemma 7.1 is that $T \setminus \bigcup_n C_n$ has at most one point. We may add this one point set to the collection $\{C_n\}$ and assume that $\{C_n\}$ covers T. $\square$

Definition 7.2. If T is a Hausdorff space we shall say that S has a G-decomposition if it can be decomposed as in Lemma 7.1. That is, we may find $\{C_n\}$ and $\{U_n\}$ satisfying one of the conditions of Lemma 7.1 and $T=\bigcup_n C_n$.

Definition 7.3 (Gruenhage [12]). An open cover $\boldsymbol{U}$ of a topological space S is σ-distributively point finite if $\boldsymbol{U}=\bigcup_n \{\boldsymbol{U}_n : n\in\mathbb{N}\}$ and for each pair of distinct points t and s in S there exists $n\in\mathbb{N}$ such that (i) there exists $U\in \boldsymbol{U}_n$ containing exactly one of t and s and (ii) s or t belongs to only finitely many elements of $\boldsymbol{U}_n$. We shall call such spaces G-spaces.

Proposition 7.4. *The following are equivalent*:

(i) *S is a G-space*;

(ii) *S has a G-decomposition such that each C_n may be taken to be the intersection of a closed and an open set*;

(iii) *S has a G-decomposition such that each C_n may be chosen in $\boldsymbol{D}(T)$.*

Proof. If S is a G-space and $\{\boldsymbol{U}_n\}$ satisfies Definition 7.3 it is easy to see that the decomposition

$$\{J(\boldsymbol{U}_n, p)\colon n\geq 1, p\geq 0\}$$

and the families $\{\boldsymbol{U}_n^p\}$, defined as above, satisfy the hypothesis of Lemma 7.1. Suppose we have (iii). For each countable family Λ of open subsets of T define $\Sigma(\Lambda)$ to be the smallest σ-algebra of sets stable under the Souslin operation containing Λ. Clearly, $\bigcup_\Lambda \Sigma(\Lambda)=\boldsymbol{D}(T)$. There exists a $\Lambda=\{W_q\colon W_q$ open and $q\in\mathbb{N}\}$ such that $\{C_n\}\subseteq\Sigma(\Lambda)$. The desired decomposition is:

$$\{W_q\cap J(\boldsymbol{U}_n, p)\colon q\geq 1, n\geq 1, p\geq 0\}.$$

The remainder follows easily from Lemma 7.1. □

We remark that any space of cardinality no greater than that of the continuum satisfies the conditions of Lemma 7.1 (see Section 9).

We say that a family of measures is constant on a set if any two measures in the family agree on the set. It is understood in the following that $\boldsymbol{M}^+(T)$ has the weak* topology.

Theorem 7.5. *Let T be compact and Φ a minimal usc compact valued map defined on the Baire space B taking values in $\boldsymbol{M}^+(T)$. Suppose that C is in $\boldsymbol{D}(T)$ and $\boldsymbol{U}=\{U_\alpha : \alpha\in\Gamma\}$ is a collection of open subsets of T such that $\{U_\alpha\cap C\colon U_\alpha\in\boldsymbol{U}\}$ is pairwise disjoint. Then $\{b\in B\colon \Phi(b)$ is constant on each $U_\alpha\cap C\}$ contains a dense G_δ subset of B.*

Proof. Define

$$W_\varepsilon = \{W\colon W \text{ is an open subset of } B \text{ and there exist a first category set } N \text{ and a finite } \Lambda \subseteq \Gamma \text{ such that for } \alpha \in \Lambda \text{ and } b \in W \setminus N,\ \Phi(b) \text{ is constant on } U_\alpha \cap C \text{ and } \mu(\textstyle\bigcup_{\beta \notin \Lambda} U_\beta) < \varepsilon \text{ for } \mu \in \Phi(b)\}.$$

Analogous to the Charting Theorem 4.3, it suffices to show that for each $\varepsilon > 0$, $\bigcup W_\varepsilon$ is dense in C. Firstly, we assume that C is a Čech complete space; it suffices at this point to assume that C is the intersection of an open and a closed set but we prove this case separately in case the reader is only interested in Čech complete spaces and he may ignore the second part of the proof. Without loss of generality we may make the following assumptions:

(i) $C = \bigcup_\alpha C_\alpha$ where $C_\alpha = C \cap U_\alpha$;

(ii) $U = T \setminus \bar{C}$;

(iii) there exists an increasing sequence $\{C_n\}$ of closed sets $((\bigcup_n C_n) \cap C) = \emptyset$ and $D \cup C = \bar{C}$ where $\bigcup_n C_n = D$;

Let W be a non empty and open subset of B and $\varepsilon > 0$. Let

$$H_1 = \{b\colon \Phi(b) \text{ is constant on each } A \text{ in the algebra generated by } \{U, C, C_n\colon n \in \mathbb{N}\}\}.$$

Then $F_1 = B \setminus H_1$ is first category. Define

$$H_{2,m} = \left\{b\colon \sup_{\mu \in \Phi(b)} \mu(\bar{C}) - \mu(C_m \cup C) < \varepsilon/8\right\}.$$

Then $B \setminus \bigcup_m H_{2,m}$ is first category. There exist $W_1 \subseteq W$ that is open and $q \in \mathbb{N}$ such that $W_1 \setminus H_{2,q} = F_2$ is first category. There exist $W_2 \subseteq W_1$, F_3 first category and real numbers r, s and t such that for $b \in W_2 \setminus F_3$ and $\mu \in \Phi(b)$ we have that

$$|\mu(U) - r| < \tfrac{1}{8}\varepsilon,$$

$$|\mu(D) - s| < \tfrac{1}{8}\varepsilon,$$

$$|\mu(C) - t| < \tfrac{1}{8}\varepsilon.$$

Fix ν in $\Phi(W_2 \setminus (F_1 \cup F_2 \cup F_3))$, choose a finite set $\Lambda \subseteq \Gamma$, choose for each $\alpha \in \Lambda$ a compact set $K_\alpha \subseteq C_\alpha$ so that $\nu(\bigcup_{\alpha \in \Lambda} K_\alpha) > t - \frac{1}{8}\varepsilon$ and choose a compact set $K_0 \subseteq U$ so that $\nu(K_0) > r - \frac{1}{8}\varepsilon$. Observe that

$$\left(K_0 \cup \left(\bigcup_{\alpha \in \Lambda} K_\alpha\right)\right) \cap \left(C_q \cup \left(\overline{\bigcup_{\alpha \notin \Lambda} C_\alpha}\right)\right) = \emptyset.$$

Let $f\colon T \to [0, 1]$ be a continuous function such that

$$f \Big| \left(K_0 \cup \left(\bigcup_{\alpha \in \Lambda} K_\alpha\right)\right) = 1,$$

$$f \Big| \left(C_q \cup \left(\overline{\bigcup_{\alpha \notin \Lambda} C_\alpha}\right)\right) = 0.$$

Thus,

$$\int f\,\mathrm{d}\nu > r+t-\tfrac{1}{4}\varepsilon.$$

The function $\mu \to \int f\,\mathrm{d}\mu$ is continuous so there exists an open set $W_3 \subseteq W_2$ so that

$$\int f\,\mathrm{d}\mu > r+s-\tfrac{1}{4}\varepsilon \quad \text{for } \mu \in W_3.$$

Let $\mu \in \Phi(W_3 \setminus (F_1 \cup F_2 \cup F_3))$. Observe that $0 \leqslant f + \chi_{C_q} \leqslant 1$ and

$$\int (f+\chi_{C_q})\,\mathrm{d}\mu \leqslant \|\mu\| < r+s+t+\tfrac{3}{8}\varepsilon.$$

Also,

$$\int (f+\chi_{C_q})\,\mathrm{d}\mu > r+s-\tfrac{1}{4}\varepsilon + \mu(C_q) > r+s-\tfrac{3}{8}\varepsilon+\mu(D) > r+s+t-\tfrac{1}{2}\varepsilon.$$

Therefore,

$$\varepsilon > \int (1-f-\chi_{C_q})\,\mathrm{d}\mu \geqslant \mu\Big(\bigcap_{\alpha \notin \Lambda} C_\alpha\Big).$$

We may find yet another first category set F_4 so that for $b \in W_3 \setminus (F_1 \cup F_2 \cup F_3 \cup F_4)$ we have that $\Phi(b)$ is constant on each C_α where $\alpha \in \Lambda$. This proves that $\bigcup W_\varepsilon$ is dense in T for each $\alpha \in \Lambda$. Retaining the same notation, we assume only that $C \in \boldsymbol{D}(T)$. Let J_i for $i=0,1$ be the points in T that belong to, respectively, none and exactly one element of U. As pointed out before, J_0 and $J_0 \cup J_1$ are closed. Recall that each element of $\boldsymbol{A}(T)$ can be written as the disjoint union of intersections of sets of the form $A = U \cap F$ where U is open and F is closed. Choose an increasing sequence $\boldsymbol{A}_n$ of finite subalgebras of $\boldsymbol{A}(T)$ such that $J_i \in \boldsymbol{A}_1$ for $i=0,1$ and C is in the smallest σ-algebra stable under the Souslin operation that contains $\bigcup \boldsymbol{A}_n = \boldsymbol{A}_\infty$. Define

$$G = \{b \in B\colon \Phi(b) \text{ is constant on } A,\ \Phi(b) \text{ is constant on each } A \in \boldsymbol{A}_\infty$$
$$\text{and } A \cap J_1 \cap U_\alpha \text{ for } A \in \boldsymbol{A}_\infty \text{ and } U_\alpha \text{ in } U\}.$$

Observe that

$$A' = \Big\{(A_0 \cap J_0) \cup \Big(\bigcup_{1 \leqslant j \leqslant l} \Big(A_j \cap J_1 \cap \Big(\bigcup_{\alpha \in \Lambda_j} U_\alpha\Big)\Big)\Big)$$
$$\cup\,(A_{l+1} \cap (T \setminus (J_0 \cup J_1)))\colon A_j \in \boldsymbol{A}_\infty, 0 \leqslant j \leqslant l+1,$$
$$\{\Lambda_j : 1 \leqslant j \leqslant l\} \text{ is a partition of } \Gamma\Big\}$$

is an algebra that contains A_∞ and for each $b \in G$, $\Phi(b)$ is constant on each $A \in A'$. Fix $b \in G$ and $\mu, \nu \in \Phi(b)$. Since μ and ν agree on A' and the class of sets where they agree is a monotone class we have that μ and ν agree on a σ-algebra Σ' containing A'. Define

$$\Sigma = \{E \,\triangle\, N: E \in \Sigma' \text{ and } (\mu+\nu)(N) = 0\}.$$

It follows from the Szpilrajn–Marczewski theorem that Σ is stable under the Souslin operation. Thus, μ and ν agree on C. □

Lemma 7.6. *Let μ and ν be nonnegative measures on the compact space T. Suppose that N and C are subsets of T and A is a countable algebra of sets such that each A in A is both μ and ν measurable and $\mu(A) = \nu(A)$. Suppose that the $\mu+\nu$ inner measure of C is $\|\mu\| + \|\nu\|$ and the $\mu+\nu$ outer measure of N is 0. If we suppose that A separates the points of $C \setminus N$ then $\mu = \nu$.*

Proof. Let $\varepsilon > 0$. Write $A = \bigcup_n A_n$ as an increasing sequence of finite algebras. Let $\{A_{n,i}\}$ be the atoms of A_n. Choose $K_{n,i}$ compact so that

$$K_{n,i} \subseteq A_{n,i} \cap (C \setminus N)$$

and

$$\sum_{n,i} |(\mu+\nu)(A_{n,i} \setminus K_{n,i})| < \tfrac{1}{2}\varepsilon.$$

Define

$$K = \bigcap_n \bigcup_i K_{n,i} \quad \text{and} \quad f_{n,i} = \chi_{K \cap K_{n,i}}.$$

Observe that $\{f_{n,i}\}$ separates the points of K and that the product of any two elements of $\{f_{n,i}\}$ is either zero or an element of $\{f_{n,i}\}$. The Stone–Weierstraß theorem says that any continuous function on K can be uniformly approximated by functions in

$$\left\{ g = \sum_i r_i f_{n,i} : r_i \in \mathbb{R},\ n \in \mathbb{N} \right\}.$$

For such a function $g = \sum_i r_i f_{n,i}$ we have that

$$\left| \int_K g \,\mathrm{d}(\mu - \nu) \right| < (\tfrac{1}{2}\varepsilon)\left(\sup_i |r_i| \right).$$

This proves that $\|\chi_K(\mu - \nu)\| < \frac{1}{2}\varepsilon$. We also have that $(\mu+\nu)(T \setminus K) < \frac{1}{2}\varepsilon$. This proves that $\|\mu - \nu\| < \varepsilon$ and the proof is complete. □

Theorem 7.7. *Suppose that T is a G-space and a subspace of the compact space K. Then*

$$M^+(T) = \{\mu \in M^+(K) \text{: the } \mu \text{ inner measure of } T \text{ is } \|\mu\|\}$$

is in C.

Proof. Let $\{C_n\}$ and $\{\boldsymbol{U}_n\}$, $\boldsymbol{U}_n = \{U_\alpha : \alpha \in \Gamma_n\}$ satisfy the definition of G-spaces. It follows from Theorem 7.5 that there exists an algebra $\boldsymbol{A}$ of subsets of T such that $C_n \cap U$ is in $\boldsymbol{A}$ for all $n \in \mathbb{N}$ and all $U \in \boldsymbol{U}_n$ and there exists a dense G_δ subset G of B such that $\Phi(b)$ is constant on each element of $\boldsymbol{A}$ for each $b \in G$. Fix $b \in G$ and $\mu, \nu \in \Phi(b)$. Define

$$N = \bigcup \{U \cap C_n : n \in \mathbb{N}, U \in \boldsymbol{U}_n, (\mu + \nu)(U \cap C_n) = 0\}.$$

It follows that $(\mu + \nu) N = 0$. There remain countable sets $\Lambda_n \subseteq \Gamma_n$ such that

$$\Lambda_n = \{\alpha \in \Gamma_n : (\mu + \nu)(U_\alpha \cap C_n) > 0\}.$$

Let $\boldsymbol{A}_1$ be a countable subalgebra of $\boldsymbol{A}$ that contains

$$\{C_n \cap U_\alpha : \alpha \in \Lambda_n\}$$

for each n. Lemma 7.6 says that $\mu = \nu$. $\square$

8. Woofs

Originally, our results were for the spaces defined in Section 7, but we have since shown, in response to a question of Jayne, that we may obtain the analogous results in what is a more general case. We would like to thank Gruenhage for pointing out that any uncountable ordinal interval is not a G-space, but do have disintegrating metrics, which is a consequence of the Radon–Nikodym property (see Section 9). The spaces under consideration here are the obvious abstraction of the Radon–Nikodym property in Banach spaces. This formal abstraction has been considered by Jayne and Rogers [17] and in a forthcoming work of Namioka [23]. Suppose T is a space and $\boldsymbol{U} = \{U_\alpha : 1 \leqslant \alpha \leqslant \gamma\}$ is a collection of open subsets of T indexed by a well ordered set, usually taken to be an interval $[1, \gamma]$ of ordinals. If $S \subseteq T$ and for every $1 \leqslant \alpha \leqslant \beta \leqslant \gamma$ we have that $S \cap U_\alpha \subseteq S \cap U_\beta$, then we shall say that $\boldsymbol{U}$ is ordered on S. Observe that if we define

$$\boldsymbol{V} = \left\{ V_\alpha : V_\alpha = \bigcup_{\beta \leqslant \alpha} U_\beta,\ U_\beta \in \boldsymbol{U} \right\}$$

then $\boldsymbol{V}$ is ordered on the entire space T and $V_\alpha \cap S = U_\alpha \cap S$ for all $\alpha \leqslant \gamma$. Given any $\boldsymbol{U} = \{U_\alpha : 1 \leqslant \alpha \leqslant \gamma\}$ that is ordered on T, by defining $V_\alpha = U_\alpha$ for $\omega \leqslant \alpha \leqslant \gamma$, $V_1 = \emptyset$, $V_n = U_{n-1}$ and $V_{\gamma+1} = T$ we may also assume that the first element is always the empty set and the last is the space T. We shall define a well ordered family of open sets such that the first element is the empty set and the last is the space T to be a woof.[1] If $\boldsymbol{U} = \{U_\alpha : 1 \leqslant \alpha \leqslant \gamma\}$ is a woof, then the atoms $\{A_\alpha : 1 \leqslant \alpha \leqslant \gamma\}$ of $\boldsymbol{U}$ are defined by $A_1 = \emptyset$ and for $1 < \alpha$

$$A_\alpha = U_\alpha \Bigg\backslash \left(\bigcup_{\beta < \alpha} U_\beta \right).$$

[1] See, *The Compact Edition of the Oxford English Dictionary*, page 3814, first definition

If U is a woof, denote by $a(U)$ the set of atoms of U. A woof U refines a woof V if $V \subseteq U$ and a woof U disintegrates a woof V if every atom of U is contained in an atom of V.

Definition 8.1. If for $i = 1, 2$ we have woofs

$$U_i = \{U_{i,\alpha_i} : \alpha_i \in \Gamma_i\}$$

then we define

$$U_1 * U_2 = \{V_{\alpha_1,\alpha_2} : (\alpha_1, \alpha_2) \in \Gamma_1 \times \Gamma_2\}$$

and

$$V_{\alpha_1,\alpha_2} = \Big(\bigcup_{\beta_1 < \alpha_1} U_{1,\beta_1}\Big) \cup \Big(U_{1,\alpha_1} \cap \Big(\bigcup_{\beta_2 \leq \alpha_2} U_{2,\beta_2}\Big)\Big).$$

Clearly, $U_1 * U_2$ is a woof ($\Gamma_1 \times \Gamma_2$ has the dictionary order) that refines U_1 and disintegrates U_2.

Proposition 8.2. *Given a sequence of woofs* $\{U_n : n \in \mathbb{N}\}$ *then there exists a sequence of woofs* $\{V_n : n \in \mathbb{N}\}$ *such that*

$$V_{n+1} \quad \textit{refines} \quad V_n,$$

$$V_n \quad \textit{disintegrates} \quad U_n.$$

Proof. Define $V_0 = \{\emptyset, T\}$ and $V_n = V_{n-1} * U_n$ for $n \in \mathbb{N}$. $\square$

Definition 8.3 [18]. A space T has a metric d that disintegrates T if for any $\varepsilon > 0$ and any $S \subseteq T$ there exists an open subset U of T such that $U \cap S \neq \emptyset$ and diam $(U \cap S) < \varepsilon$ where the diameter is in the metric d.

This next result was also obtained by Namioka [23].

Theorem 8.4. *A space T has a metric d that disintegrates T if and only if there exists a countable number of woofs that separate the points of T.*

Proof. Given the metric d choose an ordinal γ such that the cardinality of the interval $[1, \gamma]$ is greater than the cardinality of T. For each n and $\alpha \leq \gamma$ choose a closed subset $C_{n,\alpha}$ such that

$$
\begin{array}{ll}
C_{n,1} = T & \text{for all } n; \\
C_{n,\alpha} \subseteq C_{n,\beta} & \text{if } \beta < \alpha; \\
C_{n,\alpha} = \bigcap_{\beta < \alpha} C_{n,\beta} & \text{if } \alpha \text{ is a limit ordinal;} \\
C_{n,\alpha} \neq C_{n,\alpha+1} & \text{if } C_{n,\alpha} \neq \emptyset
\end{array}
$$

and

$$\operatorname{diam}(C_{n,\alpha} \setminus C_{n,\alpha+1}) < 1/n$$

Define $U_n = \{T \setminus C_{n,\alpha} : \alpha \leq \gamma\}$. This works. Assume that there exists a countable number of woofs U_n that separate the points of T and that U_{n+1} refines U_n. Let

$$\{A_{n,\alpha} : n \in \mathbb{N}, \alpha \leq \gamma\}$$

be the atoms of U_n. Define $d_n : T \times T \to \mathbb{R}$ by $d_n(s, t) = 1$ if s and t are in different atoms of U_n and $d_n(s, t) = 0$ otherwise. Clearly, d_n is a pseudometric and $d = \sum_n 2^{-n} d_n$ is a metric. For $\emptyset \neq S \subseteq T$ define

$$\alpha_n = \min\{\alpha : A_{n,\alpha} \cap S \neq \emptyset\}.$$

We have that

$$A_{n,\alpha_n} \cap S = U_{n,\alpha_n} \cap S$$

and $\operatorname{diam}(A_{n,\alpha_n} \cap S) < 2^{-n}$ which is the desired conclusion. □

Proposition 8.5. *If T is a G-space then there exist a countable number of woofs whose union constitutes a T_0 family of open sets.*

Proof. Observe that if $U = \{U_\beta : \beta \in \Gamma\}$ is any collection of open subsets of T such that

$$\{U_\beta \cap S : \beta \in \Gamma\}$$

is a partition of S then an arbitrary well ordering of Γ has the property that

$$\left\{ V_\alpha = \bigcup_{\beta \leq \alpha} U_\beta : \alpha, \beta \in \Gamma \right\}$$

is ordered and if s_1 and s_2 are in S and

$$s_1 \in V_\alpha \quad \text{if and only if} \quad s_2 \in V_\alpha$$

for all α then there exists a β such that s_1 and s_2 are both in U_β. See Section 7 for more details. □

We need a trivial result analogous to Lemma 5.6.

Lemma 8.6. *Let U be a woof in the space T and $\mu \geq 0$ a measure on T. Then μ is supported on a countable number of atoms of U.*

Proof. Let $K \subseteq T$ be compact and $\mu(K) > 0$. Choose α minimal so that $\mu(K \cap U_\alpha) > 0$. It follows from inner regularity that

$$\mu\left(K \cap \left(\bigcup_{\beta < \alpha} U_\beta\right)\right) = 0.$$

We have shown that for any compact set K of positive measure there exists an atom A so that $\mu(K \cap A) > 0$. The result follows from inner regularity. □

Theorem 8.7. *Let T be compact and Φ a minimal usc compact valued map defined on the Baire space B taking values in $M^+(T)$. Let $V=\{V_\alpha: 1\leqslant\alpha\leqslant\gamma\}$ be a woof in T and $\mathbf{B}\subseteq B(T)$ an algebra such that for each α there exists a dense G_δ set $B_\alpha\subseteq B$ such that $\Phi(b)$ is constant on $V_\alpha\cap A$ for each $A\in\mathbf{B}$, each α and each b with $b\in B_\alpha$. Let $\mathbf{A}$ be the smallest σ-algebra of subsets of T that is stable under the Souslin operation and contains $a(V)\cup\mathbf{B}$. Then*

$$\{b\in B: \Phi(b) \text{ is constant on each } A\in\mathbf{A}\}$$

contains a dense G_δ subset of B.

Proof. From monotonicity and the Szpilrajn–Marczewski theorem it suffices to verify the result for $\mathbf{A}'$, the smallest algebra of subsets of T that contains $a(V)\cup\mathbf{B}$. Observe that for $b\in\operatorname{int}\{b: 0\in\Phi(b)\}$ we have that $\Phi(b)=0$. We shall assume that for all $b\in B$ and all $\mu\in\Phi(b)$, we have that $\|\mu\|>0$. We begin by assuming that $\mathbf{B}=\{\emptyset, T\}$. It is clear that for any open set $U\subseteq T$ and any $r\in\mathbb{R}$

$$\{\mu: \mu\geqslant 0 \text{ and } \mu(U)>r\}$$

is an open subset of $M^+(T)$. Also, for an open subset O of $M^+(T)$

$$\{b\in B: \Phi(b)\subseteq O\}$$

is an open set and differs from

$$\{b\in B: \Phi(b)\cap O\neq\emptyset\}$$

by a set of the first category. For each $r>0$, define

$$M_\alpha^r=\{\mu: \mu\geqslant 0 \text{ and } \mu(V_\alpha)>r\}.$$

Define

$$B_{n,m}=\operatorname{int}\{b: \|\mu\|>(m-1)/n \quad \text{for all } \mu\in\Phi(b) \text{ and }$$
$$\Phi(b)\cap\{\nu: \|\nu\|\leqslant m/n\}\neq\emptyset\}.$$

Since Φ is minimal we have

$$B_{n,m}\subseteq\{\nu: (m-1)/n<\|\nu\|\leqslant m/n\}$$

and $\bigcup_{0\leqslant m<\infty} B_{n,m}$ is dense in B for each $n\in\mathbb{N}$. Fix $B_{n,m}$ with $m\geqslant 2$. For $0\leqslant r\leqslant (m-1)/n$ define

$$W_r=\left\{W: W \text{ is open, } W\subseteq B_{n,m} \text{ and for some } \alpha,\right.$$
$$\left.\Phi(W)\subseteq M_\alpha^r\Big\backslash\Big(\bigcup_{\beta<\alpha} M_\beta^r\Big)\right\}.$$

We need to show that $\bigcup W_r$ is dense in $B_{n,m}$ for each $0<r\leqslant(m-1)/n$. Fix an open $W_0\subset B_{n,m}$ and choose α_0 to be the minimal α such that $\Phi(W_0)\cap M_\alpha^r\neq\emptyset$. Since Φ

is minimal, there exists an open $W \subseteq W_0$ such that $\Phi(W) \subseteq M^r_{\alpha_0}$. We have proved that $\bigcup W_r$ is dense in $B_{n,m}$. The set

$$G_{n,m} = \bigcap_{\substack{0<r\leq(m-1)/n \\ r \text{ rational}}} \cup W_r$$

is a dense G_δ subset of $B_{n,m}$. Suppose that $b \in G_{n,m}$, μ, $\nu \in \Phi(b)$, and $\mu(V_\alpha) < \nu(V_\alpha)$. Choose a rational r such that

$$\mu(V_\alpha) < r < \nu(V_\alpha).$$

Since $b \in W_r$ there exist an open set W, $b \in W$ and β so that

$$\Phi(W) \subseteq M^r_\beta \Big\backslash \Big(\bigcup_{\delta<\beta} M^r_\delta \Big).$$

This is a contradiction. This proves that for $b \in G_{n,m}$ and μ, $\nu \in \Phi(b)$ then

$$|\mu(V_\alpha) - \nu(V_\alpha)| \leq 1/n$$

for all α. For $b \in G = \bigcap_n \bigcup_m G_{n,m}$, $\Phi(b)$ is constant on each V_α. Applying the results of Section 7, we know that $\Phi(b)$ is constant on each element of A'. The general case requires somewhat more complicated notation. For n, $m \geq 1$ define for each $k = 0, \ldots, m-1$

$$Q^{n,m,k}_\alpha = \bigcup \Big\{ W\colon W \text{ is open, } W \subseteq B_{n,m},\ \Phi(W) \subseteq M^{k/n}_\alpha \Big\backslash \Big(\bigcap_{\beta<\alpha} M^{k/n}_\beta \Big) \Big\}.$$

Observe that $\{Q^{n,m,k}_\alpha : 1 \leq \alpha \leq \gamma\}$ is a pairwise disjoint collection. The arguments above show that $\bigcup_\alpha Q^{n,m,k}_\alpha$ is dense in $B_{n,m}$ and for

$$b \in \bigcap_{0\leq k<m} \bigcup_\alpha Q^{n,m,k}_\alpha$$

and μ, $\nu \in \Phi(b)$ we have that $|(\mu-\nu)V_\alpha| \leq 1/n$ for all α. Suppose that

$$\emptyset \neq W = \bigcap_{0\leq j<m} Q^{n,m,j}_{\alpha_j}$$

for some choice of $\{\alpha_j : 0 \leq j < m\}$. It follows that

$$\alpha_0 \leq \alpha_1 \leq \cdots \leq \alpha_{m-1}.$$

By hypothesis there exists a dense G_δ subset $G \subseteq W$ such that for any $A \in \boldsymbol{B}$, any

$$U \in \{V_{\alpha_0}, V_{\alpha_1}, \ldots, V_{\alpha_{m-1}}, T\},$$

any $b \in G$ and any μ, $\nu \in \Phi(b)$ we have that

$$\mu(A \cap U) = \nu(A \cap U).$$

For $A \in \boldsymbol{B}$ and $j < m-1$ and $\alpha_j \leq \alpha < \alpha_{j+1}$, (or $\alpha_{m-1} < \alpha$) we have that

$$\begin{aligned} |(\mu-\nu)(V_\alpha \cap A)| &= |(\mu-\nu)(V_{\alpha_j} \cap A) + (\mu-\nu)((V_\alpha \setminus V_{\alpha_j}) \cap A)| \\ &\leq \mu((V_\alpha \setminus V_{\alpha_j}) \cap A) + \nu((V_\alpha \setminus V_{\alpha_j}) \cap A) \\ &\leq \mu(V_\alpha \setminus V_{\alpha_j}) + \nu(V_\alpha \setminus V_{\alpha_j}) \leq 2/n; \end{aligned}$$

for $\alpha < \alpha_0$ we have that

$$\mu(V_\alpha) = \nu(V_\alpha) = 0.$$

We repeat this for every choice

$$1 \leqslant \alpha_0 \leqslant \alpha_1 \leqslant \cdots \leqslant \alpha_{m-1} \leqslant \gamma.$$

Choose

$$G^{n,m}(\alpha_0, \alpha_1, \ldots, \alpha_{m-1})$$

a dense G_δ subset of

$$\bigcap_{j<m} Q^{n,m,j}_{\alpha_j}$$

so that for any

$$b \in G^{n,m}(\alpha_0, \alpha_1, \ldots, \alpha_{m-1})$$

and any μ, $\nu \in \Phi(b)$ we have that

$$|(\mu - \nu)(V_\alpha \cap E)| \leqslant 2/n$$

for all $A \in \boldsymbol{B}$ and all α. The set

$$H_{n,m} = \bigcup \{G^{n,m}(\alpha_0, \alpha_1, \ldots, \alpha_{m-1}) : 1 \leqslant \alpha_0 \leqslant \alpha_1 \leqslant \cdots \leqslant \alpha_{m-1} \leqslant \gamma\}$$

contains a dense G_δ subset of $B_{n,m}$,

$$H = \bigcap_n \bigcup_m H_{n,m}$$

contains a dense G_δ subset of B and for $b \in H$, μ, $\nu \in \Phi(b)$, $1 \leqslant \alpha \leqslant \gamma$ and $A \in \boldsymbol{B}$

$$\mu(V_\alpha \cap A) = \nu(V_\alpha \cap A).$$

It follows that for $1 \leqslant \beta \leqslant \alpha \leqslant \gamma$

$$\mu((V_\alpha \setminus V_\beta) \cap A) = \nu((V_\alpha \setminus V_\beta) \cap A).$$

The remark before Lemma 2.4 concerning the representation of $A(T)$ says that μ and ν agree on A'. $\square$

Theorem 8.8. *Let T be compact and Φ a minimal usc compact valued map defined on the Baire space B taking values in $M^+(T)$. Suppose that $\{C_p : p \in \mathbb{N}\}$ is a sequence in $D(T)$ and $U_p = \{U_{p,\alpha} : 1 \leqslant \alpha \leqslant \gamma_p\}$ is a sequence of woofs in T. Let A be the smallest σ-algebra of subsets of T that is stable under the Souslin operation and contains*

$$\bigcup_{1 \leqslant p < \infty} U_p \cup \{C_p\}.$$

Then

$$\{b \in B : \Phi(b) \text{ is constant on each } A \in A\}$$

contains a dense G_δ subset of B.

Proof. Let $\{V_p : p \in \mathbb{N}\}$ be a sequence of open sets such that each C_i is in the smallest σ-algebra containing $\{V_p\}$. Define

$$\boldsymbol{W}_p = \begin{cases} \boldsymbol{U}_{p/2} & \text{for even } p, \\ \{\emptyset, V_{(p+1)/2}, T\} & \text{for odd } p, \end{cases}$$

and define

$$\boldsymbol{V}_1 = \boldsymbol{W}_1 \quad \text{and} \quad \boldsymbol{V}_{n+1} = \boldsymbol{V}_n * \boldsymbol{W}_{n+1}.$$

There is a dense G_δ subset H_n of B such that for $b \in H_n$ and $\mu, \nu \in \Phi(b)$, we have that μ and ν agree on $\boldsymbol{V}_n$. For fixed $b \in \bigcap_n H_n$ and $\mu, \nu \in \Phi(b)$, we have that μ and ν agree on $\bigcup_n a(\boldsymbol{V}_n)$; since $\boldsymbol{V}_n \subset \boldsymbol{V}_{n+1}$ they agree on the smallest σ-algebra $\boldsymbol{A}'$ stable under the Souslin operation containing $\bigcup_n a(\boldsymbol{V}_n)$. Define

$$\boldsymbol{A}'' = \{A \triangle N : A \in \boldsymbol{A}' \text{ and } (\mu + \nu)(N) = 0\}.$$

From Definition 8.1 and Lemma 8.6 we have that

$$\bigcup_n \boldsymbol{W}_n \subseteq \boldsymbol{A}''.$$

Thus, $\boldsymbol{A} \subseteq \boldsymbol{A}''$ because $\boldsymbol{A}''$ is stable under the Souslin operation. The proof is complete. □

Theorem 8.9. *Denote by $\boldsymbol{D}$ the class of completely regular spaces such that T is in $\boldsymbol{D}$ if and only if T has a countable number of woofs that separate its points. If T is in $\boldsymbol{D}$, then*

(i) *$M^+(T)$ is in $\boldsymbol{D}$ and*
(ii) *any perfect image of T is in $\boldsymbol{D}$.*

Proof. If T is in $\boldsymbol{D}$ let $\{\boldsymbol{U}_n\}$ be a sequence of woofs that separate the points of T. We may assume that $\boldsymbol{U}_{n+1}$ refines $\boldsymbol{U}_n$. Let $[1, \gamma_n]$ be a sequence of ordinal intervals so that

$$\boldsymbol{U}_n = \{U_{n,\alpha} : 1 \leq \alpha \leq \gamma_n\}.$$

If we choose an ordinal $\gamma > \gamma_n$ for all n and define $U_{n,\alpha} = T$ for $\alpha > \gamma_n$ then we may assume that $\gamma_n = \gamma$ for all n. For each rational $r \geq 0$, each $n \in \mathbb{N}$ and each $\alpha < \gamma$ define

$$M_{n,\alpha}^r = \{\mu : \mu \geq 0, \mu(T) > r, \mu(U_{n,\alpha}) > r\}$$

and

$$M_{n,\gamma}^r = M^+(T).$$

Clearly, for each n and r

$$\boldsymbol{W}_{n,r} = \{M_{n,\alpha}^r : \alpha \leq \gamma\}$$

is a woof. Suppose that μ and ν are in $M^+(T)$ and

$$\mu \in M^r_{n,\alpha} \quad \text{if and only if} \quad \nu \in M^r_{n,\alpha}.$$

This implies that $\mu(U_{n,\alpha}) = \nu(U_{n,\alpha})$ for all n and all α. Therefore, μ and ν agree on the smallest σ-algebra containing $\bigcup_n U_n$. The remainder of the details are exactly like those of Section 7. Suppose that $p: T \to S$ is perfect onto. Define

$$V_{n,\alpha} = \{s \in S: p^{-1}(s) \subseteq U_{n,\alpha}\}$$

and

$$V_n = \{V_{n,\alpha} : 1 \leq \alpha \leq \gamma_n\}.$$

Suppose that s_1 and s_2 are in S and

$$p^{-1}(s_1) \subseteq U_{n,\alpha} \quad \text{if and only if} \quad p^{-1}(s_2) \subseteq U_{n,\alpha}.$$

Define

$$\alpha_n = \min\{\alpha: p^{-1}(s_1) \subseteq U_{n,\alpha}\}.$$

Since each $p^{-1}(s_i)$ is compact it follows that

$$\emptyset \neq K_{i,n} = p^{-1}(s_i) \cap \left(\bigcap_{\beta < \alpha_n} T \setminus U_{n,\beta} \right).$$

Since U_{n+1} refines U_n it follows that for each i and each n

$$K_{i,n+1} \subseteq K_{i,n}.$$

Choose $t_i \in \bigcap_n K_{i,n}$. Observe that for each n,

$$t_1, t_2 \in U_{n,\alpha_n} \setminus \left(\bigcup_{\beta < \alpha_n} U_{n,\beta} \right).$$

For each n, t_1 and t_2 belong to the same atom; this proves that $t_1 = t_2$ and

$$p^{-1}(s_1) \cap p^{-1}(s_2) \neq \emptyset$$

and this completes the proof of (ii). $\square$

The following combines the results given here with those of [35].

Theorem 8.10. *Let T be a space that is k-analytic and has a countable number of woofs that separate its points. Suppose that K is compact, $S \subseteq K$ and $f: T \to S$ is continuous and onto. Then $M(S)$, the space of (signed) measures on K supported on S, belongs to C.*

Proof. We know that $M^+(T)$ is in C. It is shown in [35] that $M^+(T)$ is also k-analytic, when considered as a subset of $M^+(\beta T)$. Define

$$\psi : \mathbb{R}^2 \times (M^+(T))^2 \to M(\beta T)$$

by

$$\psi(a, b, \mu, \nu) = a\mu + b\nu.$$

The image A of ψ is also k-analytic and in C. Denote also by f the canonical extension of f from βT to K. The canonical operator from $M(\beta T)$ to $M(K)$ carries A onto $M(S)$ which must also be in C (details are in [35]). □

Theorem 8.11. *Let X be a Banach space and K a weak* compact and convex subset of X^*. Suppose $f: T \to X^*$ is weak* continuous, $E \subseteq f(T)$ where E is the set of extreme points of K and T is k-analytic with a countable number of woofs that separate its points. Then K is in C.*

Theorem 8.12. *If a space T is in D and is a Baire space then it contains a dense G_δ subset that is homeomorphic to a metrizable Baire space.*

Proof. Let d be the metric defined as above with respect to the countable number of woofs $U_n = \{U_{n,\alpha}\}$ and let $\{A_{n,\alpha}\}$ be the associated atoms. It is routine that

$$G = \bigcap_n \bigcup_\alpha \operatorname{int} A_{n,\alpha}$$

is a dense G_δ subset on which the given topology and the topology determined by d agree. □

Theorem 8.13. *A space T is in C (respectively, in D) if and only if there exists an open cover $\{U_\alpha\}$ of T such that each U_α is in C (respectively, in D). A space T is in C (respectively, in D) if and only if there exists a sequence $\{T_n\}$ in $D(T)$ that covers T such that each T_n is in C (respectively, in D).*

Theorem 8.14. *Suppose Λ is a set, X_λ is a Banach space and X_λ^* in its weak* topology is in D for each $\lambda \in \Lambda$. Suppose $1 < p < \infty$. Then if $Y = (\sum_\lambda X_\lambda)_{\ell_p}$ we have that Y^* is in D. Also, if $Y = (\sum_\lambda X_\lambda)_{c_0}$ then Y^* is in D.*

Proof. Define

$$Y = \left(\sum_\lambda X_\lambda\right)_{\ell_1}.$$

We shall show that

$$Z = \{(x_\lambda^*) \in Y^* : (\|x_\lambda^*\|) \in c_0(\Lambda)\}$$

in the weak* topology of Y^* is in $\boldsymbol{D}$. For each λ let $U_{\lambda,n}=\{U_{\lambda,n,\alpha}\}$ be a sequence of woofs that separate the points of X_λ^* and for each rational number r we shall denote by $W_{r,\lambda}$ the subset of Z defined by

$$\{(x_\xi^*)\in Z\colon \|x_\lambda^*\|\geqslant r\}.$$

Assume that Λ is well ordered. For each $n\in\mathbb{N}$, each rational r and each $(\lambda,\alpha)\in\Lambda\times\Gamma$ define

$$\tilde{U}_{\lambda,n,\alpha}=U_{\lambda,n,\alpha}\times\prod_{\beta\neq\lambda}X_\beta^*$$

and

$$V_{r,\lambda,n,\alpha}=\left(\bigcup_{\xi<\lambda}W_{r,\xi}\cup(W_{r,\lambda}\cap\tilde{U}_{\lambda,n,\alpha})\right).$$

We put the dictionary order on $\Lambda\times\Gamma$. The remainder of the proof is routine. □

9. Examples

It is much easier to prove something about the class $\boldsymbol{C}$ than to determine whether or not a particular topological space is in $\boldsymbol{C}$. For the remainder of the paper see [35, 7, 26, 27, 25] for definitions and bibliography. Let $\boldsymbol{X}$ be the class of all Banach spaces such that X^* (in the weak* topology) is in $\boldsymbol{C}$. At this point we know that if X is in $\boldsymbol{X}$ then any subset T of X^* that has a tactic in the weak* topology has a dense subspace M that is homeomorphic to a complete metric space.

In this section we shall denote by $\boldsymbol{D}$ the class of Hausdorff spaces that have a countable number of woofs that separate points. As we have seen, in several different ways, metric spaces are in $\boldsymbol{D}$. Indeed, it is easy to see that developable spaces are in $\boldsymbol{D}$; hence, all Moore spaces, a class which contains many unusual examples as well as being a class for which there are important undecidable propositions, are also in $\boldsymbol{D}$ (see [25] for definitions and background material).

The class $\boldsymbol{C}$ was designed to contain the following spaces: T is a topological space that is homeomorphic to a subset of X^*, X^* has the weak* topology, where X is a Banach space such that X^* has the Radon–Nikodym property [35]. In fact, $\boldsymbol{D}$ contains hereditarily dentable subsets of Banach spaces. Namioka has characterized compact spaces homeomorphic to subsets of dual Banach spaces having the Radon–Nikodym property as those compact spaces having a lower semicontinuous fragmenting metric (defined to be Radon–Nikodym compact). If a compact space T has no measures except purely atomic measures then it is a Radon–Nikodym compact. Let $T=[1,\omega_1]$ be the first uncountable ordinal interval with the order topology. Clearly, ω_1 is in the closure of $[1,\omega_1)$ but is not the limit of a sequence in $[1,\omega_1)$, which shows that T is not a Corson compact. The Banach space $C(T)$ is an Asplund space because T supports no non atomic inner regular Borel measures.

As pointed out to us by Gruenhage and others it is quite easy to see that T is not a G-space. We shall briefly outline this fact. Suppose T admits a decomposition satisfying (i) to (iii) above. Define

$$S_n = T_n \setminus \left(\bigcup \{ U \in \mathcal{U}_n \} \right)$$

and partition $\mathbb{N}$ as follows:

(i) $N_1 = \{ n \in \mathbb{N} \colon T_n \text{ is countable} \}$;

(ii) $N_2 = \{ n \in \mathbb{N} \colon n \notin N_1,\ S_n \text{ is countable} \}$; and

(iii) $N_3 = \mathbb{N} \setminus (N_1 \cup N_2)$.

Choose $\beta < \omega_1$ such that

$$\beta > \sup\left(\left(\bigcup_{n \in N_1} T_n \right) \cup \left(\bigcup_{n \in N_2} S_n \right) \right).$$

For each $n \in N_2$, $\mathcal{U}_n$ is an open cover by pairwise disjoint sets of the uncountable compact space $T_n \cap [\beta, \omega_1]$ so there exists $U_n \in \mathcal{U}_n$ such that

$$T_n \cap [\beta, \omega_1] \cap U_n$$

is closed and uncountable. Thus,

$$C = \left(\bigcap_{n \in N_2} (T_n \cap [\beta, \omega_1] \cap U_n) \right) \cap \left(\bigcap_{n \in N_3} S_n \right)$$

is an uncountable subset of $[1, \omega_1]$ and

$$\{ T_n \cap U \colon n \in \mathbb{N} \text{ and } U \in \mathcal{U}_n \}$$

does not separate the points of C.

If Γ is a set we shall denote by $\Pi(\Gamma)$ the Γ-fold product of $[0, 1]$ and $\Sigma(\Gamma) \subseteq \Pi(\Gamma)$ is that subset of $\Pi(\Gamma)$ consisting of countably supported elements. We recall that a Corson compact is a compact space that is homeomorphic to a subset of $\Sigma(\Gamma)$ for some Γ. The canonical examples of spaces that are not in $\mathcal{C}$ are $\Pi(\Gamma)$ where Γ is any uncountable set. It is a result of Corson (see [25]) that if K is a Corson compact then $C(K)$ is Lindelöf in the simple topology.

The dictionary square has cardinality of the continuum but is not in the class $\mathcal{C}$ (see [34]). Any space of cardinality no greater than that of the continuum satisfies the hypothesis of Lemma 7.1.

An amalgamation of examples of Haydon, Talagrand and Kunen (see [25] for complete details and references) shows that, assuming the continuum hypothesis, there exists a Corson compact K such that K contains no dense metric space; thus $C(K)$ is not in $\mathcal{X}$. An example is given in [44] of a Corson compact that has no dense metrizable subspace using no special axioms of set theory. An example is given in [12] of a Gruenhage space that is not a Corson compact. There is a separable compact space which is the union of a countable number of Eberlein compacts but is not metrizable, hence not a Gul'ko compact. Pol [27] has an example of a separable compact dispersed space K (hence $C(K)^*$ has the Radon–Nikodym property and $C(K)$ is in $\mathcal{X}$) but $C(K)$ is not Lindelöf in the simple topology. Interestingly, the example of Pol is different from that of James described below because the space X contains a complemented subspace isomorphic to the separable Hilbert space. If K is dispersed, then $C(K)$ contains no non-trivial reflexive subspaces. For more details about the next two theorems see [38] and its references.

Theorem 9.1 (James [16], see also [20]). *There exists a separable Banach space X such that* (i) X^* *is not separable and* (ii) X^{**} *has the Radon–Nikodym property.*

Theorem 9.2. *If we define $Z = X^*$, where X is space of James as defined above, then the Banach space Z has the properties that Z^* has the Radon–Nikodym property* (*thus Z is in $\mathcal{X}$*) *but Z is not isomorphic* (*linearly homeomorphic*) *to any subspace of $C(K)$ if $C(K)$ is Lindelöf in the simple topology. It follows that there exists a compact space K such that $C(K)$ is in $\mathcal{X}$ but $C(K)$ is not Lindelöf in the simple topology.*

As we have remarked, the class $\mathcal{C}$ was defined so that Banach spaces whose duals have the Radon–Nikodym property and weakly compactly generated (in particular, reflexive) Banach spaces are in $\mathcal{X}$. Debs [6] proved a result much stronger than the fact that weakly k-analytic spaces are in $\mathcal{X}$. Gruenhage [12] used the work of Gul'ko [13] and Sokolov [31] to analyze the unit ball of X^* in its weak* topology if the Banach space X in its weak topology is the upper semicontinuous image of a separable metric space. An obvious consequence of this work (see also [45, 21]) is, using our terminology, that such Banach spaces belong to our $\mathcal{X}$. Fundamental to the work of Vasak, Gul'ko and Sokolov is the theorem of Amir and Lindenstrauss [1].

10. Gul'ko compact spaces

Amir and Lindenstrauss [1] proved that a weakly compact subset of a Banach space is weakly homeomorphic to a subspace of some $c_0(\Gamma)$ for some Γ; hence, weakly compact spaces (Eberlein compact spaces) are Corson compact. Gul'ko [13] proved that if $C(K)$ is weakly countably determined then K is a Corson compact (see also [45, 21, 24, 42]). Surprisingly short proofs of these results can be found in [40].

Definition 10.1 [45]. A Banach space X is weakly countably determined (wcd) if there exists a sequence $\{K_n\}$ of weak* compact subsets of X^{**} such that for every $x \in X$ there exists $\xi \in \mathbb{N}^{\mathbb{N}}$ so that

$$x \in \bigcap_{n=1}^{\infty} K_{\xi(n)} \subseteq X.$$

The following is known and can be found, for example, in [40].

Lemma 10.2. *Let X be a Banach space. The following are equivalent:*

(i) *X is weakly countably determined;*

(ii) *there exists a separable metric space M and an usc compact (in the weak topology) valued map Φ defined on M such that $X = \bigcup_{t \in M} \Phi(t)$;*

(iii) *there exists a separable metric space M and a map $\Phi : M \to 2^X$ such that $\Phi(F)$ is relatively weakly compact for any compact subset F of M and $\bigcup_{t \in M} \Phi(t)$ separates the points of X^* and*

(iv) *if $X = C(K)$, the space of continuous functions on the compact space K, there exists a separable metric space M and a map $\Phi : M \to 2^X$ such that $\Phi(F)$ is bounded and relatively compact in the simple topology for any compact subset F of M and $\bigcup_{t \in M} \Phi(t)$ separates the points of K.*

Definition 10.3. A compact space K such that $C(K)$ is weakly countably determined will be called a Gul'ko compact or a Vasak compact.

If T is a completely regular space then vT will denote the Hewitt real compactification of T (see [7]). A boundary point of vT is a point in $vT \setminus T$. Since the defining property of vT is that each real valued continuous function on T has a unique extension to vT we shall make no notational distinction between a continuous $f : T \to \mathbb{R}$ and its extension to vT. If T is completely regular, then a subset E of T is bounding if every continuous real valued function on T is bounded on E; clearly, this is equivalent to E being relatively compact in vT. Lindelöf spaces are real compact (see also [11]).

Theorem 10.4. *Suppose that T is completely regular and $\Phi : M \to \not{k}(T)$ is a map from the separable metric space M such that*

(i) *$\bigcup_{m \in M} \Phi(m) = T$; and*

(ii) *$\bigcup_{m \in K} \Phi(m)$ is bounding for every compact $K \subseteq M$.*

Then vT is the upper semicontinuous compact valued image of a separable metric space.

Proof. For each $f \in C(T)$ and each $m \in M$ define

$$\lambda(f, m) = \lim_{m' \to m} \sup_{t \in \Phi(m')} f(t)$$

which is finite. Define $\Psi : M \to \not{k}(vT)$ by

$$\Psi(m) = \bigcap_f \{s \in vT : -\lambda(-f, m) \leq f(s) \leq \lambda(f, m)\}.$$

Clearly, $\Psi(m)$ is compact. Given any $m \in M$, any $f \in C(T)$ and any $\alpha > \lambda(f, m)$ there exists a neighborhood W of m such that $f(t) < \alpha$ for $t \in \Psi(m')$ and $m' \in W$. Since any neighborhood of $\Psi(m)$ is determined by a finite number of functions

$\{f_i\}$ and a finite set $\{\alpha_i\}$ with $\alpha_i > \lambda(f_i, m)$ we have proved that Ψ is usc. Thus, the image $\Psi(M)$ of Ψ is Lindelöf and

$$T \subseteq \Psi(M) \subseteq vT.$$

Since vT is the smallest real compact space containing T it follows that $\Psi(M) = vT$. □

For a bibliography concerning the following, see [31]. The equivalence of (i), (iv) and (v) below follow from the above and are due to us.

Proposition 10.5. *Let T be a completely regular space. The following are equivalent:*

(i) *there exists a separable metric space M and an upper semicontinuous compact valued map $\Phi: M \to \mathscr{k}(T)$ such that $T = \bigcup_{m \in M} \Phi(m)$;*

(ii) *there exists a sequence of subsets $\{T_n : n \in \mathbb{N}\}$ of T such that if U is any open cover of T and $I = \{N \in \mathbb{N}$: T_n is covered by a finite subset of $U\}$ then $T = \bigcup_{n \in I} T_n$;*

(iii) *the same as* (ii) *with the additional assumption that each T_n is closed;*

(iv) *T is real compact and there exist a separable metric space M and a mapping $\Phi: M \to \mathscr{k}(T)$ with the properties that $T = \bigcup_{m \in M} \Phi(m)$ and $\Phi(K)$ is relatively compact for each compact subset K of M; and*

(v) *T is Lindelöf and there exist a separable metric space M and a mapping $\Phi: M \to \mathscr{k}(T)$ with the properties that $T = \bigcup_{m \in M} \Phi(m)$ and $\Phi(K)$ is relatively countably compact for each compact subset K of M.*

Proof. It is easy to see that (i) implies that T is Lindelöf. We begin by showing that (ii) implies (i). For each $t \in T$ define

$$I(t) = \{n : t \in T_n\},$$

$$K(t) = \bigcap_{n \in I(t)} \overline{T_n}$$

$$G = \{(t, \xi) : (t, \xi) \in T \times \mathbb{N}^{\mathbb{N}} \text{ and } \xi(\mathbb{N}) = I(t)\}.$$

We need to show that each $K(t)$ is compact and if $K(t) \subseteq W$ where W is open then there exists $\Lambda \subseteq I(t)$ that is finite such that

$$K(t) \subseteq \bigcap_{n \in \Lambda} \overline{T_m} \subseteq W.$$

Fix some $t \in T$, let U be any open cover of $K(t)$, and let V be any open cover of $K(t)$ such that for each $V \in V$ there exists a $U \in U$ such that $\bar{V} \subseteq U$. By hypothesis there exists $m \in I(t)$ such that T_m has a finite subcover from $V \cup \{T \backslash K(t)\}$. Then $\overline{T_m}$ has a finite subcover from $U \cup \{T \backslash K(t)\}$ and $K(t)$ has a finite subcover from U. Suppose that $K(t) \subseteq W$ where W is open. Since $K(t)$ is compact there exists U open such that

$$K(t) \subseteq U \subseteq \bar{U} \subseteq W.$$

Then

$$W = \{T \backslash \overline{(T_m \cap (T \backslash U))}: m \in I(t)\}$$

is an open cover of T. There exist $\Lambda \subseteq I(t)$ finite and $q \in I(t)$ such that T_q is covered by

$$\{T \backslash \overline{(T_m \cap (T \backslash U))}: m \in \Lambda\}.$$

This means that

$$T_q \cap \left(\bigcap_{m \in \Lambda} T_m \right) \subseteq U.$$

It follows immediately that (i) implies (iv) and (v). Suppose that we have (iv). Let $\{W_n : n \in \mathbb{N}\}$ be a basis for the topology of M and let $T_n = \overline{\Phi(W_n)}$. Define $I(t)$ and $K(t)$ as above. With nearly the same details as above, it can be shown that each $K(t)$ is compact and if $K(t) \subseteq W$ where W is open then there exists $\Lambda \subseteq I(t)$ that is finite such that

$$K(t) \subseteq \bigcap_{n \in \Lambda} \overline{T_m} \subseteq W.$$

It follows that we have (iii) and that T is Lindelöf, from which (v) follows. □

Definition 10.6. We shall call spaces satisfying Proposition 10.5 Lindelöf Sigma spaces and denote them by L-Σ spaces.

Corollary 10.7. *The completely regular continuous image of a L-Σ space is a L-Σ space. The countable union of L-Σ spaces is a L-Σ space. The countable product of L-Σ spaces is a L-Σ space. A closed subspace of a L-Σ space is a L-Σ space.*

Clearly, the second condition below is equivalent to: there exists a bounded subset T of $C(K)$ that may be decomposed $T = \bigcup_{n \in \mathbb{N}} T_n$ such that for any $k \in K$ if we define

$$I_k = \{n \in \mathbb{N}: \{f \in T_n : f_n(k) \neq 0\} \text{ is finite}\}$$

then $T = \bigcup_{n \in I_k} T_n$. A consequence of the following is that a Gul'ko compact is a Gruenhage compact. A prototype for the following result is in [30].

Theorem 10.8 (Gul'ko, Sokolov). *Let K be a compact space. Then $C(K)$ is weakly countably determined if and only if there exists $U = \bigcup_n U_n$ a T_0 family of open F_σ subsets such that for any $k \in K$ if*

$$I = \{m: k \text{ is in only finitely many elements of } U_m\}$$

then $U = \bigcup_{m \in I} U_m$.

Proof. Let T be a subset of the unit ball of $C(K)$ that separates the points of K, $\{f \in T: \int f \, d\mu \neq 0\}$ is countable for each $\mu \in C(K)^*$ and, for convenience, we assume that the zero function is in T. By including $f \vee 0$ and $(-f) \vee 0$ we may assume that $0 \leq f \leq 1$ for all $f \in T$. Let

$$R : C(K) \to c_0(\Delta)$$

be a continuous and one to one linear function. Denote by $\{e_\delta : \delta \in \Delta\}$ and $\{e_\delta^* : \delta \in \Delta\}$ the respective canonical bases of $c_0(\Delta)$ and $\ell_1(\Delta)$. For each $f \in T$, let $\Delta(f) \subseteq \Delta$ be the support of $R(f)$; that is

$$\Delta(f) = \{\delta \in \Delta : e_\delta^*(R(f) \neq 0\}.$$

Fix $f_1 \in T$ and let

$$S_1 = \{f \in T : \Delta(f) \cap \Delta(f_1) \neq \emptyset\}.$$

Since $R^*(e_\delta^*)$ is only countably nonzero on T we have that S_1 is countable. Let $\Delta_1 = \bigcup_{f \in S_1} \Delta(f)$ and define

$$S_2 = \{f \in T : \Delta(f) \cap \Delta_1 \neq \emptyset\}$$

which is also a countable set. We construct countable sets $S_n \subseteq T$ and $\Delta_n \subseteq \Delta$ such that for $f \in S_n$ we have that $\Delta(f) \subseteq \Delta_n$ and

$$S_{n+1} = \{f \in T : \Delta(f) \cap \Delta_n \neq \emptyset\}.$$

Let $T_\theta = \bigcup_n S_n$ and $\Delta_\theta = \bigcup_n \Delta_n$. Then we have that $f \in T_\theta$ if and only if $\Delta(f) \subseteq \Delta_\theta$ if and only if $\Delta(f) \cap \Delta_\theta \neq \emptyset$ for each $f \in T$. We may define partition $T = \bigcup_\theta T_\theta$ and $\Delta = \bigcup_\theta \Delta_\theta$ into disjoint sets this way. Since each T_θ is countable, let $T_n \subseteq T$ such that $T_n \cap T_\theta$ has at most one point and $T = \bigcup_{n \in \mathbb{N}} T_n$. We assume that the zero function is in each T_n. Observe that, for each n, $\{R(f) : f \in T_n\}$ is a disjointly supported and bounded set that contains the zero vector, thus a weakly compact subset of $c_0(\Delta)$. Since R is one to one it follows that each T_n is a weakly closed subset of $C(K)$ and a L-Σ space in the weak topology. Let $T_n = \bigcup_m T_{n,m}$ be sets that satisfy Proposition 10.5(ii). Let $0 < r < 1$ be rational and let

$$U_{r,n,m} = \{\{f > r\} : f \in T_{n,m}\}$$

and let

$$U = \{\{f > r\} : f \in T_{n,m},\ n, m \in \mathbb{N}, \text{and } 0 < r < 1 \text{ is rational}\}.$$

Clearly, U is a T_0 family of open F_σ sets. Fix $k \in K$ and let

$$I = \{(r, n, m) : k \text{ is in only finitely many elements of } U_{r,n,m}\}.$$

Fix $n \in \mathbb{N}$ and a rational $0 < r < 1$. If $f \neq 0$ is in T_n and we choose any δ in $\Delta(f)$ we have that $R^*(e_\delta^*)(f) \neq 0$ and $R^*(e_\delta^*)(g) = 0$ for all other g in T_n. Thus, each non zero element of T_n is an open subset of T_n. We shall define a weak open cover $V = \{V_0, V_1, \dots\}$ of T_n in the following way: let

$$V_0 = \{f \in T_n : f(k) < r\} \cup \{f \in T_n : f(k) = r\}$$

and V_i be one point subsets of T_n such that

$$\bigcup_{j \in \mathbb{N}} V_j = \{f \in T_n : f(k) > r\}.$$

By definition $T_n = \bigcup \{T_{n,m} : T_{n,m}$ has a finite cover from $V\}$. If $T_{n,m}$ has a finite cover from V then (r, n, m) is in I. We have proved that

$$U = \bigcup_{(r,n,m) \in I} U_{r,n,m}.$$

Suppose that $U = \bigcup_n U_n$ is a T_0 family of open F_σ sets such that for any $k \in K$ if

$$I = \{m : k \text{ is in only finitely many elements of } U_m\}$$

then $U = \bigcup_{m \in I} U_m$. Analogously, assume that the empty set is in each U_n. For each $U \in U$ let $f_U : K \to [0, 1]$ be a continuous function such that $\{f \neq 0\} = U$ and define $T_n = \{f_U : U \in U_n\}$. For each finite $\kappa \subseteq \mathbb{N}$ let $T_\kappa = \bigcap_{n \in \kappa} T_n$. Let $F \subseteq K$ be finite and

$$V = \{f \in C(K) : |f(k)| < \varepsilon \text{ for all } k \in F\}.$$

For each $k \in F$ let $I(k) = \{m : U_m \text{ is finite at } k\}$. Let

$$J = \{\kappa : \kappa \cap I(k) \neq \emptyset \text{ for all } k \in F\}.$$

Clearly, $\bigcup_n T_n = \bigcup_{\kappa \in J} T_\kappa$. Also, $T_\kappa \setminus V$ is a finite set for each $\kappa \in J$. From Proposition 10.5 we know that $\bigcup_\kappa T_\kappa$ is a L-Σ space and separates the points K, thus $C(K)$ is weakly countably determined. $\square$

11. Some remarks

As we well know by now, a Baire space is a topological space such that no non empty open subset is first category. A barely Baire space, terminology for which we are not responsible, it being found in [8], is a Baire space T such that there exists another Baire space S such that $T \times S$ is not a Baire space. Oxtoby, assuming the continuum hypothesis, showed that there exists a metrizable barely Baire space, and Paul Cohen, as one of the earliest applications of forcing, showed that such an example must exist within the usual axioms. See Fleissner and Kunen [8] for a bibliography, where a method is given for constructing metrizable barely Baire spaces of any density character greater than ω_1. Suppose that M is a Baire metrizable space and suppose that M is a Baire property set in some compactification. A particular case of this, according to the Szpilrajn–Marczewski theorem, is that there exist a compact space T and $S \subseteq T$ that is homeomorphic to M and S is in the smallest σ-algebra containing the open sets and stable under the Souslin operation.

Suppose that $M = U \triangle F$ in some compactification T where U is open and F is first category. We assume that M is dense in T. It follows that U is dense in T. Suppose that $\{T_n\}$ is a sequence of closed and nowhere dense sets such that $F \subseteq \bigcup_n T_n$. Then $U \setminus \bigcup_n T_n$ is Čech complete and a dense subset of M. A Čech complete and metrizable space is homeomorphic to a complete metric space. This proves that M is not barely Baire (see, for example [46]). Thus, a barely Baire metrizable space is not a Baire property set in any compactification. Our class $\boldsymbol{C}$ contains the metric spaces, thus it does contain barely Baire spaces, and from the results of Section 6 we know that any barely Baire space in $\boldsymbol{C}$ cannot be a Baire property set in some compact space. This indicates that the structure of the class of non metrizable spaces in the class $\boldsymbol{C}$ could be even more complicated than the metrizable subclass.

Independently, the idea of refining woofs, using a different vocabulary, was used in [29]. A result similar to our Theorem 8.9 for compact spaces appears in [29]: the techniques there seem to require further refinement to work in the case of non compact spaces. One interesting observation of [29] is that in a regular space T one can refine a woof in a stronger sense. Suppose that we have a woof $\boldsymbol{U} = \{U_\alpha : \alpha \leq \gamma\}$ in a regular space T. Let $\boldsymbol{V} = \{V_\alpha : \alpha \leq \gamma\}$ be a well ordering of all the open subsets of T (for convenience, we assume that the index sets are the same). Define

$$W_{\alpha_1,\alpha_2} = \Big(\bigcup_{\beta<\alpha_1} U_{\beta_1}\Big) \cup \Big(U_{\alpha_1} \cap \Big(\bigcup_{\substack{\delta \leq \alpha_2 \\ V_\delta \subseteq U_{\alpha_1}}} V_\delta\Big)\Big)$$

and

$$\boldsymbol{W} = \{W_{\alpha_1,\alpha_2} : (\alpha_1, \alpha_2) \in \Gamma \times \Gamma\}.$$

The woof $\boldsymbol{W}$ defined in this way has the property that if A is an atom of $\boldsymbol{W}$ then $\bar{A}$ is contained in an atom of $\boldsymbol{U}$. Suppose that T is a Čech complete space that has a sequence of woofs $\boldsymbol{U}_n$ that separate its points. Let T_n be an increasing sequence of compact subsets of βT such that $\beta T \setminus (\bigcup_n T_n) = T$. Define

$$\boldsymbol{V}_n = \{V : V \text{ is open in } T \text{ and } \bar{V}^{\beta T} \cap T_n = \emptyset\}.$$

Combine our previous results with the observation above to produce a sequence of woofs $\boldsymbol{W}_n$ such that $\boldsymbol{W}_{n+1}$ refines $\boldsymbol{W}_n$ and if A is an atom of $\boldsymbol{W}_n$ then $\bar{A} \subseteq V$ for some V in $\boldsymbol{V}_n$. If we take the metric defined by the sequence $\boldsymbol{W}_n$ then this metric d has two additional properties: it generates a topology finer than that of the given topology on T and T is complete in this metric. By using the results of [17, 41] we obtain the following: suppose that B is a complete metric space (completeness is not necessary, because one can extend to completions by means of Lavrentieff type techniques [35]) and $\Phi : B \to \mathscr{k}(T)$ is upper semicontinuous and compact valued then there exists a (single valued) function $f : B \to T$ that is in the first Baire class with respect to the metric d and $f(b) \in \Phi(b)$ for all $b \in B$.

Acknowledgement

We are grateful to J. Jayne and I. Namioka for conversations and criticisms during the preparation of this material. Part of this material was given in lectures in Murcia in 1987 and 1988. Also, the audience for these lectures in Linz was long suffering.

References

[1] D. Amir and J. Lindenstrauss, The structure of weakly compact sets in Banach spaces, Ann. of Math. 88 (1968) 35-46.
[2] N. Bourbaki, Topologie Général, Actualités Sci. Indust. (Hermann, Paris) 1942-1949.
[3] G. Choquet, Lectures in Analysis (Benjamin, New York, 1969).
[4] M.M. Čoban and P.S. Kenderov, Generic Gâteaux differentiability of convex functionals in $C(T)$ and the topological properties of T, Math. and Math. Education (1986) 141-149.
[5] M.M. Day, Normed Linear Spaces (Springer, Berlin, 1973).
[6] G. Debs, Espaces k-analytiques et espaces de Baire de fonctions continues, Mathematika 32 (1985) 218-228.
[7] R. Engelking, General Topology (PWN, Warsaw, 1977).
[8] W.G. Fleissner and K. Kunen, Barely Baire spaces, Fund. Math. CI (1978) 239-240.
[9] D. Fremlin, Čech analytic spaces, Note University of Essex, 8 December, 1980.
[10] Z. Frolík, Baire spaces and some generalizations of complete metric spaces, Czechoslovak Math. J. 20 (1970) 406-467.
[11] L. Gillman and M. Jerison, Rings of Continuous Functions (Princeton Univ. Press, Princeton, NJ, 1962).
[12] G. Gruenhage, A note on Gul'ko compact spaces, to appear.
[13] S.P. Gul'ko, On the structure of spaces of continuous functions and their complete paracompactness, Russian Math. Surveys 34 (1979) 36-44.
[14] J. Hagler and C. Stegall, Banach spaces whose duals contain complemented subspaces isomorphic to $C[0, 1]^*$, J. Funct. Anal. 13 (1973) 233-251.
[15] R. Haydon, Some more characterizations of Banach spaces containing ℓ_1, Proc. Camb. Phil. Soc. 80 (1976) 269-276.
[16] R.C. James, A separable somewhat reflexive Banach space with non-separable dual, Bull. Amer. Math. Soc. 80 (1974) 738-743.
[17] J.E. Jayne and C.A. Rogers, K-analytic sets, in: C.A. Rogers, ed., Analytic Sets (Academic Press, London, 1980).
[18] J.E. Jayne and C.A. Rogers, to appear.
[19] J.L. Kelley, General Topology (Van Nostrand, Princeton, NJ, 1955).
[20] J. Lindenstrauss and C. Stegall, Examples of separable spaces which do not contain ℓ_1 and whose duals are nonseparable, Studia Math. 54 (1975) 81-105.
[21] S. Mercourakis, On weakly countably determined Banach spaces, Trans. Amer. Math. Soc. 300 (1987) 307-327.
[22] I. Namioka, Separate and joint continuity, Pacific J. Math. 51 (1974) 515-531.
[23] I. Namioka, Radon-Nikodym compact spaces and fragmentability, Mathematika 34 (1987) 258-291.
[24] I. Namioka and R.F. Wheeler, Gul'ko's proof of the Amir-Lindenstrauss theorem, Contemp. Math. 52 (1986) 113-120.
[25] S. Negrepontis, Banach Spaces and Topology, in: J. Vaughan and K. Kunen, Eds., Handbook of Set Theoretic Topology (North-Holland, Amsterdam, 1984).
[26] R. Pol, On pointwise and weak topology in function spaces, University of Warsaw, 1984.
[27] R. Pol, Concerning function spaces on separable compact spaces, Bull. Acad. Pol. Sci. (1977) 993-998.
[28] J.D. Pryce, A device of R.J. Whitely applied to pointwise compactness in spaces of continuous functions, Proc. London Math. Soc. 23 (1971) 532-536.

[29] N.K. Ribarska, Internal characterization of fragmentable spaces, Mathematika 34 (1987) 243-257.
[30] H. Rosenthal, The heredity problem for weakly compactly generated Banach spaces, Compositio Math. 28 (1974) 83-111.
[31] G.A. Sokolov, On some classes of compact spaces lying in Σ-products, Comm. Math. Univ. Carolinae 25 (1984) 219-231.
[32] C. Stegall, A class of topological spaces and differentiation of functions in Banach spaces, Vorlesungen aus dem Fachbereich Mathematik der Universität Essen, 1983.
[33] C. Stegall, Gâteaux differentiation of functions on a certain class of Banach spaces, Functional Analysis: Surveys and Recent Results III (North-Holland, Amsterdam, 1984).
[34] C. Stegall, More Gâteaux differentiability spaces, in: Banach Spaces, Proceedings, Missouri 1984, Springer Lecture Notes 1166 (Springer, Berlin, 1985).
[35] C. Stegall, Applications of descriptive topology in functional analysis, Universität Linz, 1985.
[36] C. Stegall, The Radon-Nikodym property in conjugate Banach spaces, Trans. Amer. Math. Soc. 206 (1975) 213-223.
[37] C. Stegall, The duality between Asplund spaces and spaces with the Radon-Nikodym property, Israel J. Math. 29 (1978) 408-412.
[38] C. Stegall, Topological spaces with dense subspaces that are homeomorphic to complete metric spaces and the classification of $C(K)$ Banach spaces, Mathematika 34 (1987) 101-107.
[39] C. Stegall, Generalizations of a theorem of Namioka, Proc. Amer. Math. Soc. 102 (1988) 559-564.
[40] C. Stegall, A proof of the theorem of Amir and Lindenstrauss, Israel J. Math. 68 (1989) 185-192.
[41] C. Stegall, Functions of the first Baire class with values in Banach spaces, Proc. Amer. Math. Soc., to appear.
[42] M. Talagrand, Espaces de Banach faiblement k-analytique, Ann. of Math. 110 (1979) 407-438.
[43] M. Talagrand, Espaces de Baire et espaces de Namioka, Math. Ann. 270 (1985) 159-164.
[44] S. Todorčević, Trees and linearly ordered sets, in: Handbook of General Topology (North-Holland, Amsterdam, 1984).
[45] L. Vasak, On one generalization of weakly compactly generated Banach spaces, Studia Math. 70 (1981) 11-19.
[46] H.E. White, Topological spaces that are α-favorable for a player with perfect information, Proc. Amer. Math. Soc. 50 (1975) 477-482.